Quantum Chaotic Features of Black Holes in Brickwall Model

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Quantum Gravity of Open Systems, 05 February 2025

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Reference

hep-th cond-mat.stat-mech gr-qc nlin.CD quant-ph

Brickwall One-Loop Determinant: Spectral Statistics & Krylov Complexity

Hyun-Sik Jeong, Arnab Kundu, Juan F. Pedraza arXiv:2412.12301

Submitted 16 December, 2024; **originally announced** December 2024.

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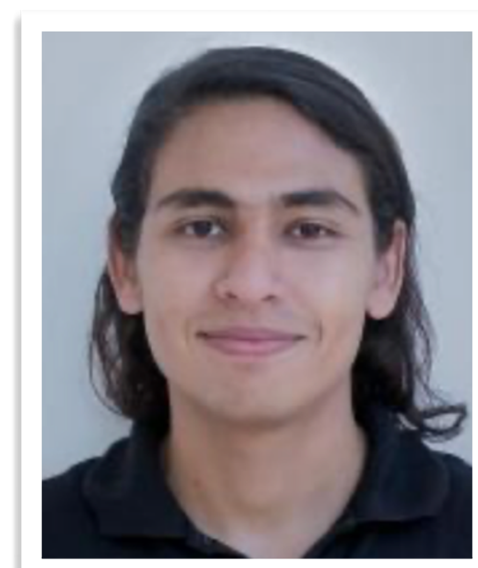
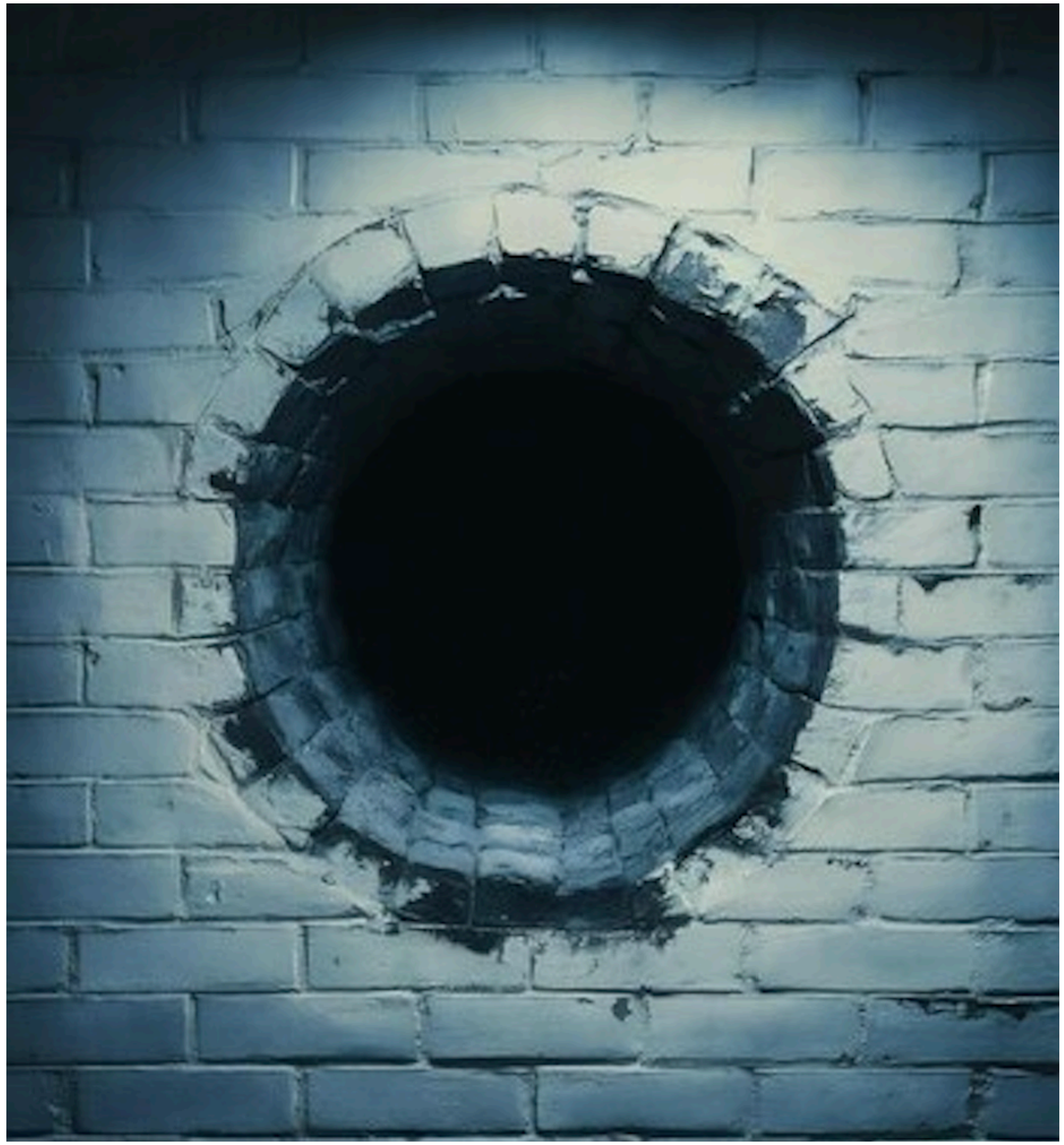


Juan F. Pedraza

Motivation



Brickwall model for (AdS) BH

[hep- th/9802109, hep- th/9802150]

GKP-Witten Formula

(Fundamental formula of AdS/CFT)

$$Z_{\mathrm{QFT}} = Z_{gravity}$$

(d-dim field theory)

(d+1 dim gravity theory)

GKP-Witten Formula

(Fundamental formula of AdS/CFT)

$$\left\langle e^{\int \phi^0 \mathcal{O}} \right\rangle_{\mathrm{QFT}} = e^{S_{gravity}[\phi|_{z=0} = \phi^0]}$$

Source

Operator

[hep- th/9802109, hep- th/9802150]

GKP-Witten Formula

(Fundamental formula of AdS/CFT)

$$\left\langle e^{\int \phi^0 \mathcal{O}} \right\rangle_{QFT} = e^{S_{gravity}[\phi|_{z=0} = \phi^0]}$$

Classical
on-shell action

[hep- th/9802109, hep- th/9802150]

GKP-Witten Formula

(Fundamental formula of AdS/CFT)

$$\left\langle e^{\int \phi^0 \mathcal{O}} \right\rangle_{QFT} = e^{S_{gravity}[\phi|_{z=0} = \phi^0]}$$

Near AdS Boundary: $\phi(x,z) = \phi^0(x) + \phi^1(x)\, z + \mathcal{O}(z^2)$

$z \to 0$

Bulk Fields

Source

Response

Power of GPKW Formula

(N-point functions)

$$\langle \mathcal{O}(x_1) \ldots \mathcal{O}(x_n) \rangle = \frac{\delta^{(n)} \mathcal{S}^{\text{ren}}_{\text{gravity}}[\phi]}{\delta\phi_0(x_1) \ldots \delta\phi_0(x_n)} \Big|_{\phi_0=0}$$

Linear response theory (n=2)

↓

Hard Computation

Transport for strongly interacting field theory

Power of GPKW Formula

(N-point functions)

$$\langle \mathcal{O}(x_1) \ \dots \ \mathcal{O}(x_n) \rangle \ = \ \frac{\delta^{(n)} \, \mathcal{S}^{\text{ren}}_{\text{gravity}}[\phi]}{\delta\phi_0(x_1) \ \dots \ \delta\phi_0(x_n)} \Big|_{\phi_0=0}$$

Linear response theory (n=2)

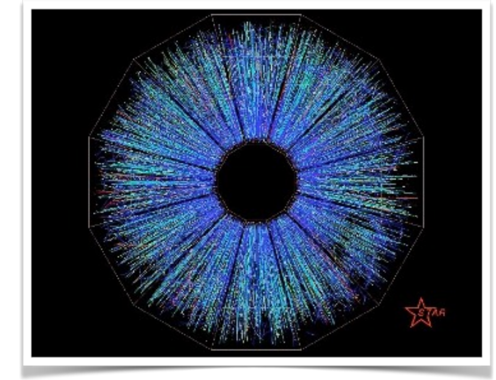
Hard Computation

Easy Computation

Transport for strongly interacting field theory

Shear Viscosity

EX 1: Quark-Gluon Plasma



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Viscosity in Strongly Interacting Quantum Field Theories from Black Hole Physics

P. K. Kovtun, D. T. Son, and A. O. Starinets

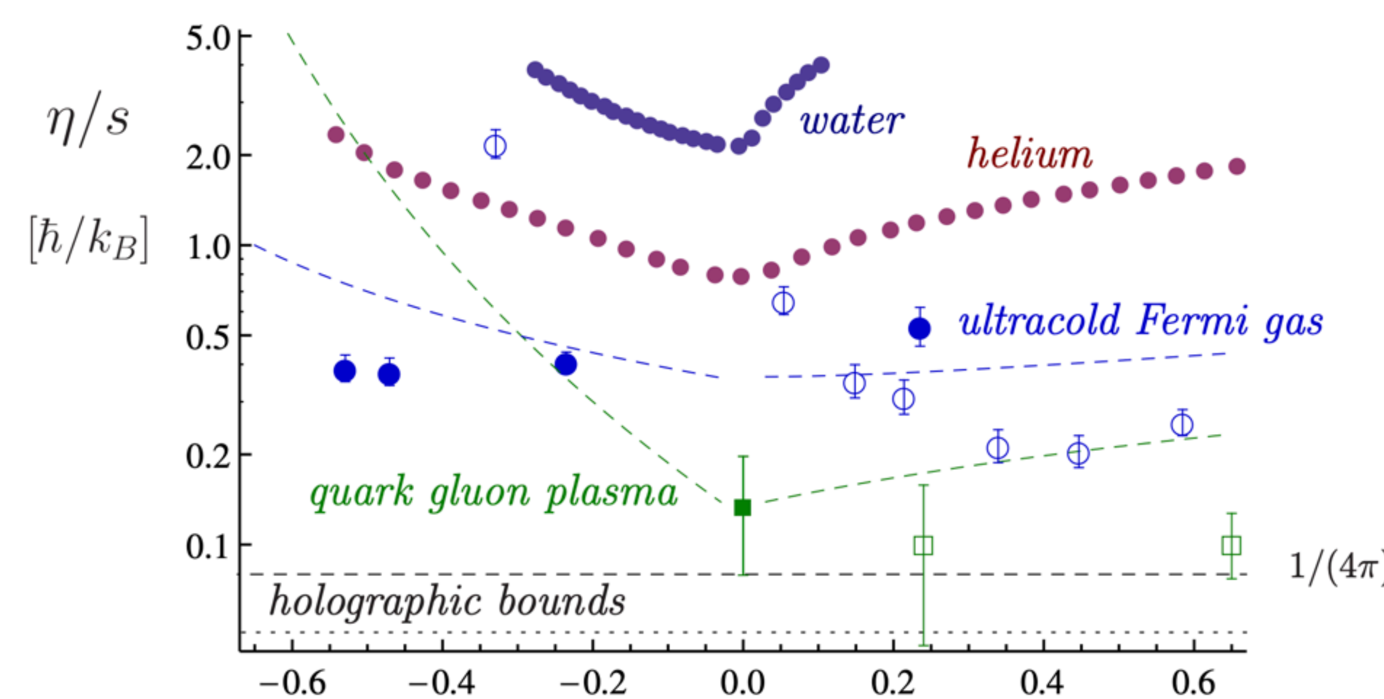
Phys. Rev. Lett. **94**, 111601 – Published 22 March 2005

- Viscosity of Quark-Gluon Plasma (Insights into the early universe / heavy-ion collisions)

Perturbative Theory: $\eta/s = \dfrac{A}{\lambda^2 \log(B/\sqrt{\lambda})} \gg 1$ (small t' Hooft coupling λ)

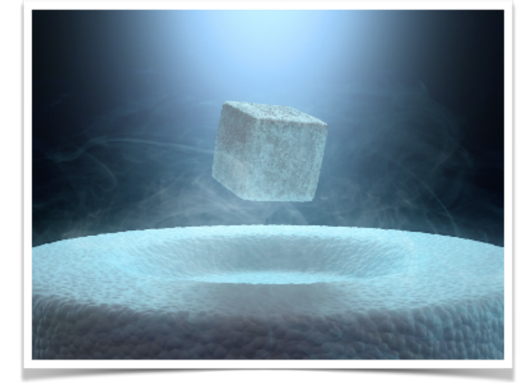
Gravity Theory: $\eta/s = 1/(4\pi)$

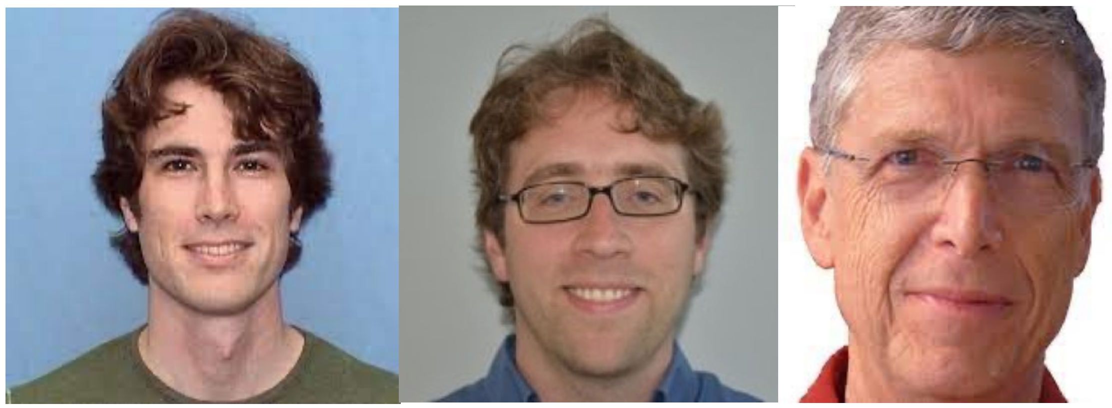
Experiment: $\eta/s \approx 0.19$



Electric Conductivity

EX 2: Superconductors





PHYSICAL REVIEW LETTERS

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Building a Holographic Superconductor

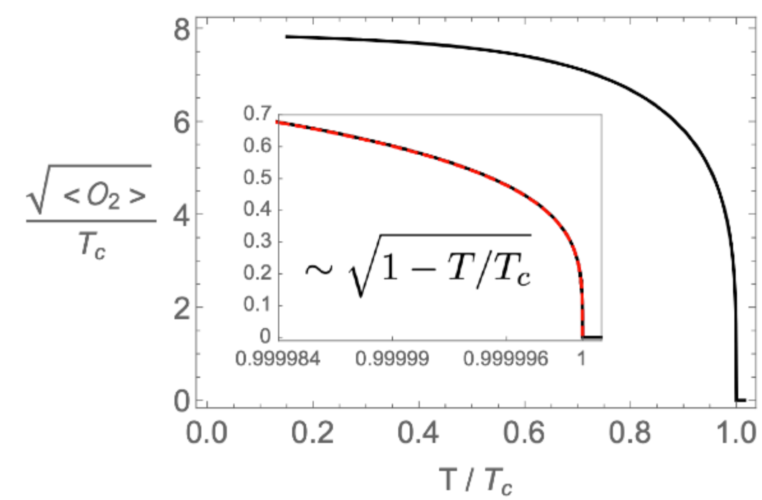
Sean A. Hartnoll, Christopher P. Herzog, and Gary T. Horowitz

Phys. Rev. Lett. **101**, 031601 – Published 14 July 2008

- **New perspective of high-Tc superconductivity**

- **Spontaneous condensation, infinite conductivities, ⋯**

$$\sigma(\omega) = \sigma_0 + \left(\frac{i}{\omega} + \delta(\omega)\right)\frac{\rho_s}{\mu}$$



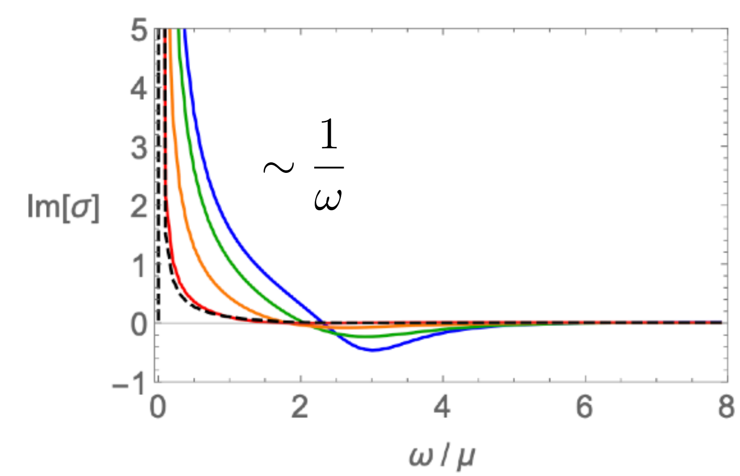
- **Energy gap**

BCS Theory: $\omega_g/T_c \approx 3.5$ (Weakly-coupled BCS Theory)

Gravity Theory: $\omega_g/T_c \approx 8$ (Holographic Superconductors)

Experiment: $\omega_c/T_c \approx 7.9 \pm 0.5$ (**High-Tc Cuprates**)

Nature **447**, 569–572 (2007)



AdS boundary conditions

Near AdS Boundary: $$\phi(x,z) = \phi^0(x) + \phi^1(x)\, z + \mathcal{O}(z^2)$$

$z \to 0$

Bulk Fields

Source

Response

fixing

Dirichlet boundary condition

(Standard quantization)

AdS boundary conditions

Near AdS Boundary: $\phi(x, z) = \phi^0(x) + \phi^1(x)\, z + \mathcal{O}(z^2)$

$z \to 0$

Bulk Fields

Source

Response

fixing

[Witten, Marolf, Ross, ...]

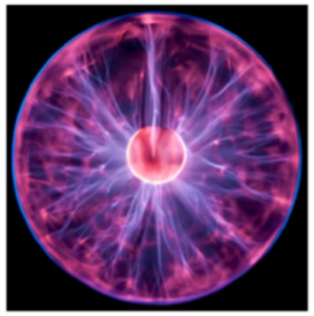
Mixed boundary condition

Dirichlet / Neumann / Robin

AdS boundary conditions

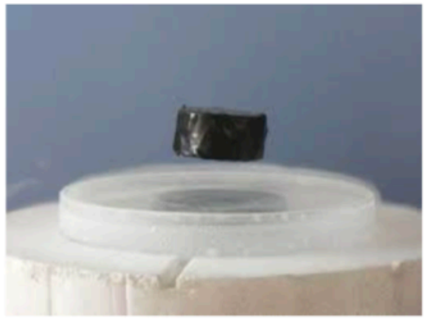
(Power of Mixed Boundary conditions)

electromagnetism



plasmas

plasmons



superconductors

Dynamical gauge fields

Electromagnetic interactions

Coulomb interactions,

...

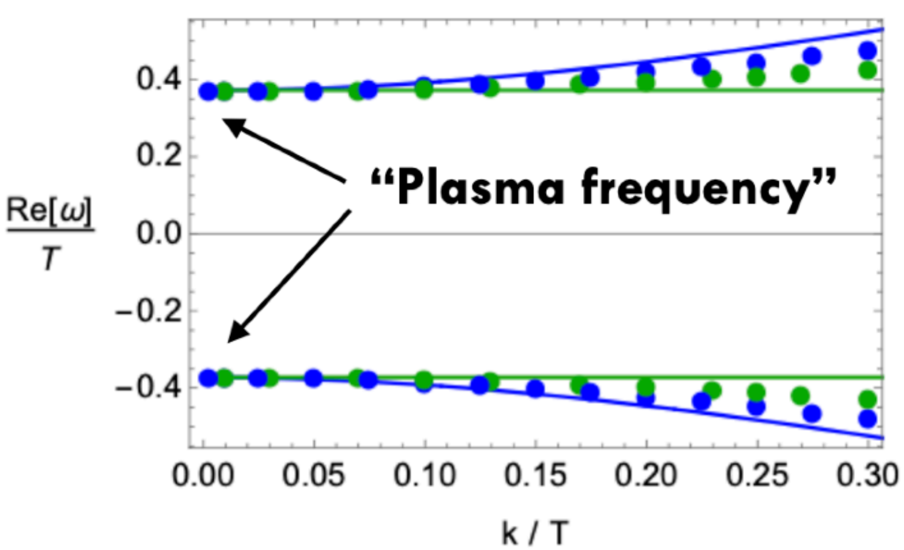
Applied AdS/CFT to Realistic System

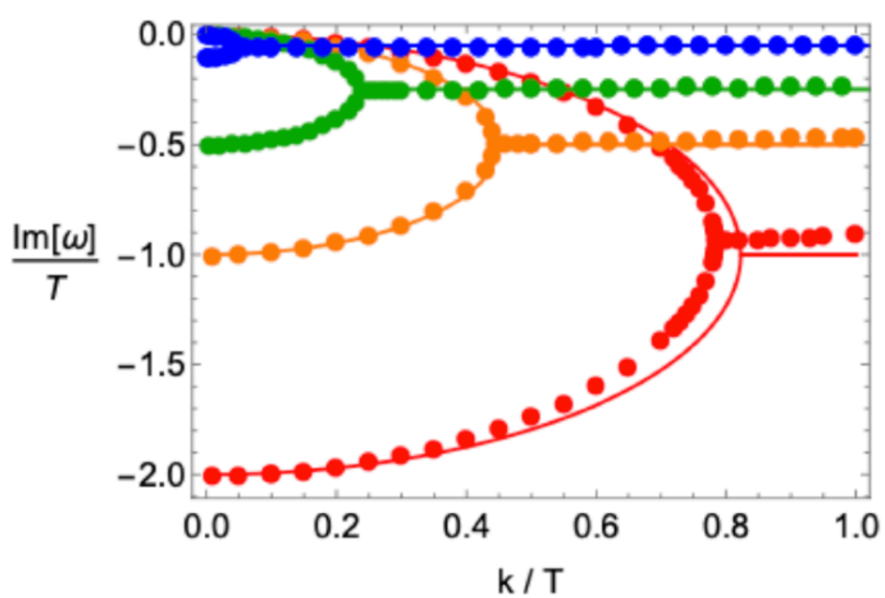
EFTs
vs. Poles

Examples

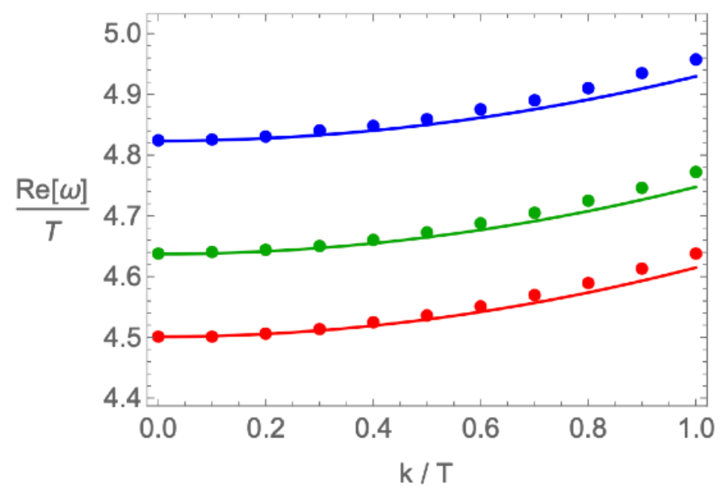
(Collective Excitations = Quasi-Normal Modes)

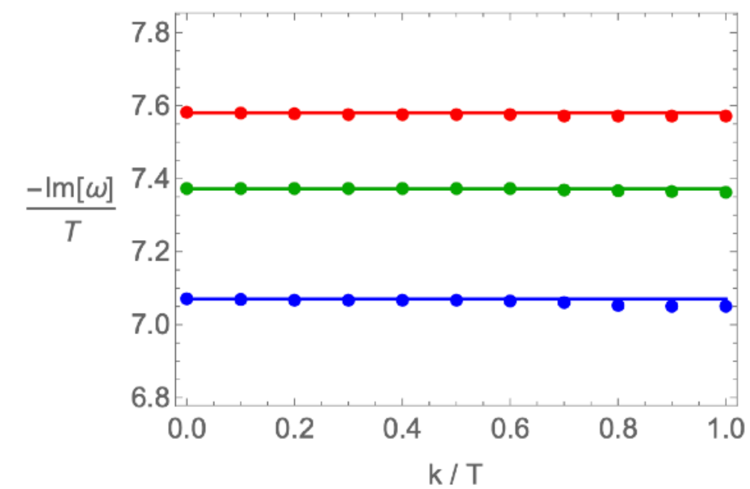
Magneto-hydrodynamics



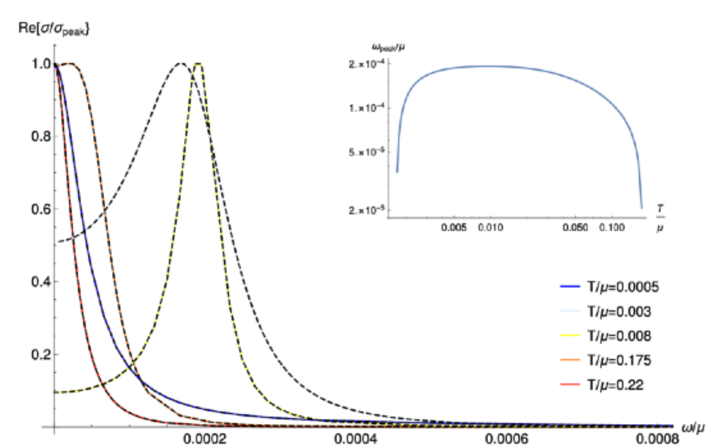


Anderson-Higgs mechanism (GL theory)



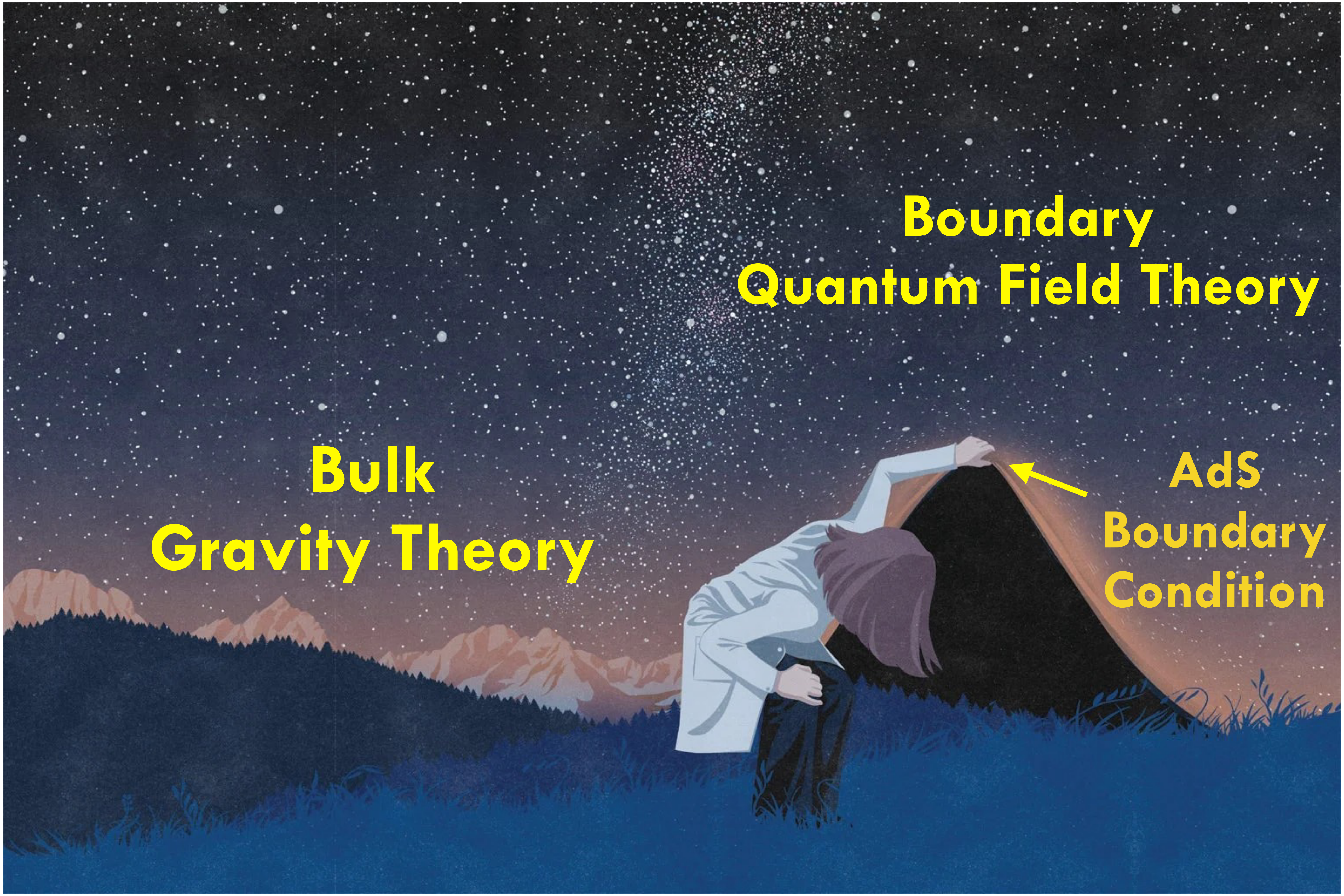


Charge density waves



[Many people]

…


Boundary
Quantum Field Theory
Bulk
Gravity Theory
AdS
Boundary
Condition

Horizon Conditions

Effect of "relaxed" horizon conditions in AdS/CFT?

Any interesting physics in black holes?

Implications on boundary physics?

Horizon Conditions

This is THE black hole

Nothing comes out
from the horizon ⋯

physically natural ⋯

retarded 2-pt fn ⋯

blabla ⋯

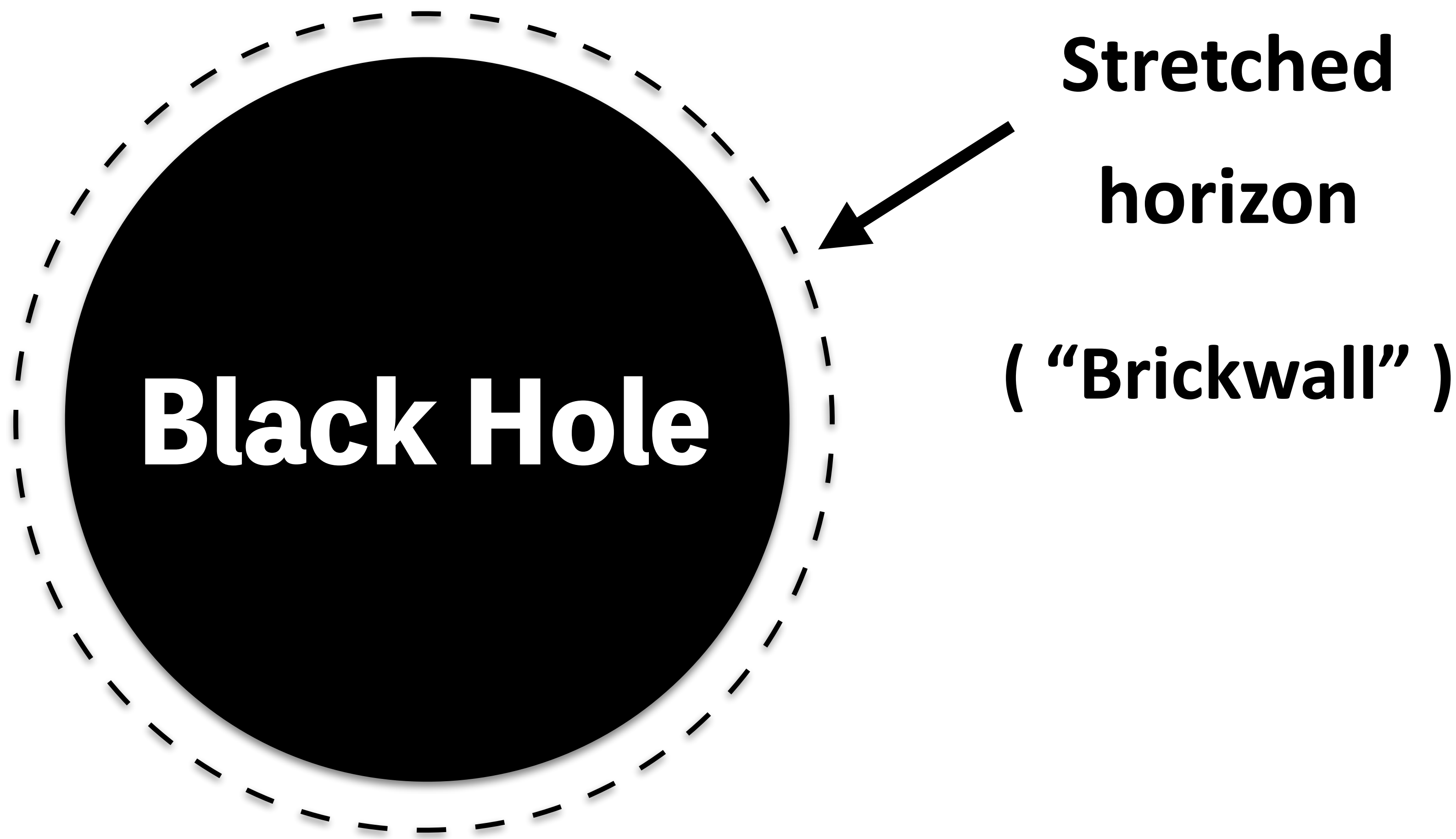
Incoming
boundary condition

Horizon Conditions

How about
the “stretched" horizon, then?

[Nucl. Phys. B 256 (1985) 727,
Phys. Rev. D 48 (1993) 3743]

Brickwall model



At a technical level, it's the brickwall model from 't Hooft

A Dirichlet wall is placed *ad hoc* outside the event horizon

[Nucl. Phys. B 256 (1985) 727,
Phys. Rev. D 48 (1993) 3743]

Brickwall model

Simple yet effective model to capture some aspects of quantum black holes

- Quantization of the probe scalar fields: Partition function, free energy

(quantized energy spectrum, semi-classical calculations)

[Nucl. Phys. B 256 (1985) 727]

- Fine-grained information: reproducing the entropy as a one-loop effect

(up to a numerical pre-factor)

- Beyond flat spacetime: AdS geometry [Nucl. Phys. B 895 (2015) 1]

(Including nice reviews of t' Hooft's calculations)

Brickwall model in AdS BH

- Quantization of the probe scalar fields: **energy spectrum "statistics"**

Synthetic fuzzballs: a linear ramp from black hole normal modes

Suman Das (Saha Inst.), Chethan Krishnan (Bangalore, Indian Inst. Sci.), A. Preetham Kumar (Bangalore, Indian Inst. Sci.), Arnab Kundu (Saha Inst.) (Aug 31, 2022)

Published in: *JHEP* 01 (2023) 153 • e-Print: 2208.14744 [hep-th]

Fuzzballs and random matrices

Suman Das (Saha Inst.), Sumit K. Garg (Manipal U.), Chethan Krishnan (Bangalore, Indian Inst. Sci.), Arnab Kundu (Saha Inst.) (Jan 27, 2023)

Published in: *JHEP* 10 (2023) 031 • e-Print: 2301.11780 [hep-th]

Brickwall in rotating BTZ: a dip-ramp-plateau story

Suman Das (HBNI, Mumbai), Arnab Kundu (HBNI, Mumbai and CERN) (Oct 10, 2023)

Published in: *JHEP* 02 (2024) 049 • e-Print: 2310.06438 [hep-th]

...

Brickwall model in AdS BH

- Quantization of the probe scalar fields: **energy spectrum "statistics"**

When a stretched horizon is close to the event horizon energy spectrum can exhibit **quantum chaos** signature consistent with random matrix theory

1. Level spacing distribution of Gaussian Unitary Ensemble (GUE)

2. Dip-Ramp-Plateau structure with a linear lamp in the SFF

(Spectral Form Factor)

Brickwall model in AdS BH

Probe **Scalar** Field	GUE	GOE	GSE
Level spacing distribution	✔		
Spectral Form Factor	✔		
Krylov Complexity			

Our Work

[2412.12301]

1. Random matrix theory across various ensembles
2. Modern tool of quantum chaos: Krylov complexity
3. Dynamics of probe Fermion field

• OUTLINE

1 | Preliminaries: chaos diagnostics

: level spacing distribution, SFF, Krylov complexity

2 | Normal modes in Brickwall model

: energy spectrum from scalar/fermion fields

3 | Results

: chaotic features of black holes

Preliminaries: chaos diagnostics

: level spacing distribution, SFF, Krylov complexity

[J. Math. Phys. 3 (1962)]

[Phys. Rev. Lett. 52 (1984)]

Random Matrix Theory

- A pivotal role in identifying universal features of quantum chaotic systems

- Three main classes in RMT (collections of RM with specific properties) (different Gaussian measures)

Gaussian Unitary Ensemble (GUE)	**Gaussian Orthogonal Ensemble (GOE)**	**Gaussian Symplectic Ensemble (GSE)**
Hermitian matrices	Real symmetric matrices	Hermitian quaternionic matrices
Invariant under U. conjugation	Invariant under O. conjugation	Invariant under S. conjugation
Systems lacking time-reversal symm.	Systems with time-reversal symm.	Systems with time-reversal symm. (no rotational symm.)

[J. Math. Phys. 3 (1962)]

[Phys. Rev. Lett. 52 (1984)]

Bohigas-Giannoni-Schmit (BGS) Conjeture

- The central conjecture in the study of quantum chaos

: postulates that the fine structure of **the energy spectrum of a quantum chaotic Hamiltonian** can be approximated **by the statistical behavior of RMT**

Level Statistics

$$H_{??} \longleftrightarrow H_{\mathrm{GUE}},\ H_{\mathrm{GOE}},\ H_{\mathrm{GSE}}$$

(Some chaotic Hamiltonian)

[J. Math. Phys. 3 (1962)]

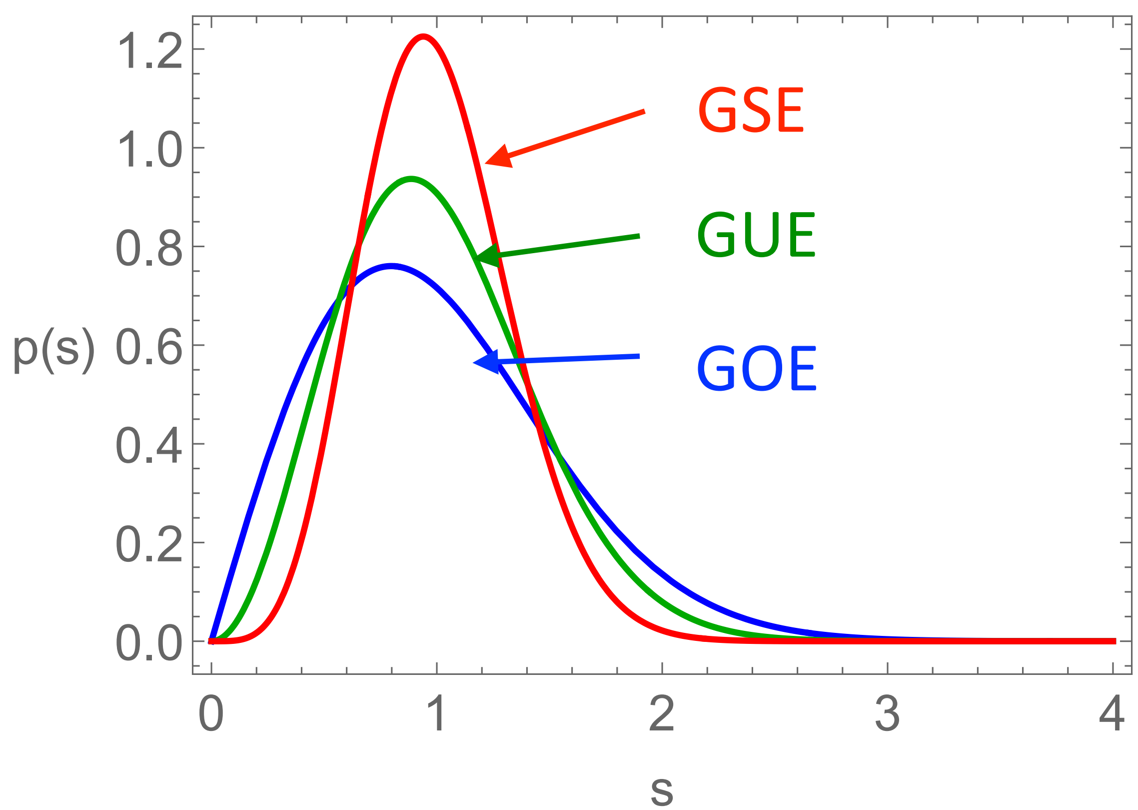
[Phys. Rev. Lett. 52 (1984)]

Bohigas-Giannoni-Schmit (BGS) Conjeture

- **Level spacing distribution** (the probability of finding two adjacent energy levels)

$$p_{\mathrm{GOE}} = \frac{\pi}{2} s\, e^{-\frac{\pi}{4}s^2}, \qquad p_{\mathrm{GUE}} = \frac{32}{\pi^2} s^2\, e^{-\frac{4}{\pi}s^2}, \qquad p_{\mathrm{GSE}} = \frac{2^{18}}{3^6\pi^3} s^4\, e^{-\frac{64}{9\pi}s^2}$$

: level repulsion signifies that, in **chaotic systems**, energy levels tend to avoid clustering



[J. Math. Phys. 3 (1962)]

[Phys. Rev. Lett. 52 (1984)]

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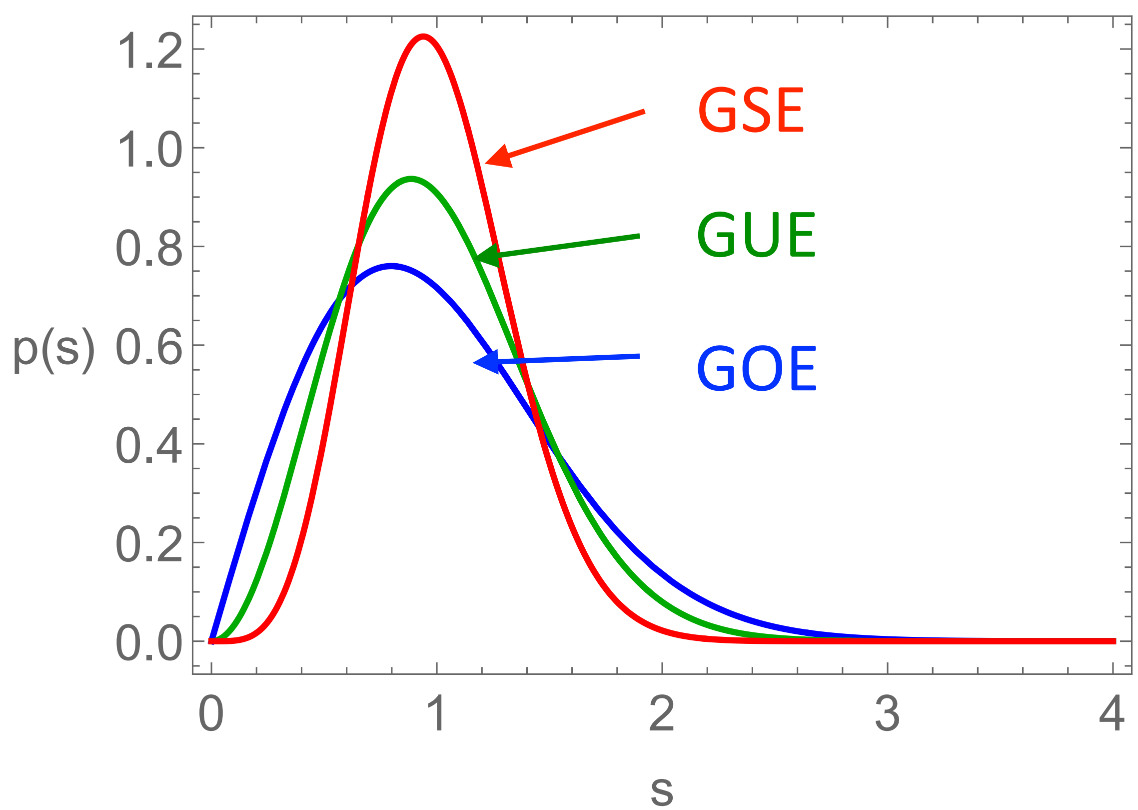
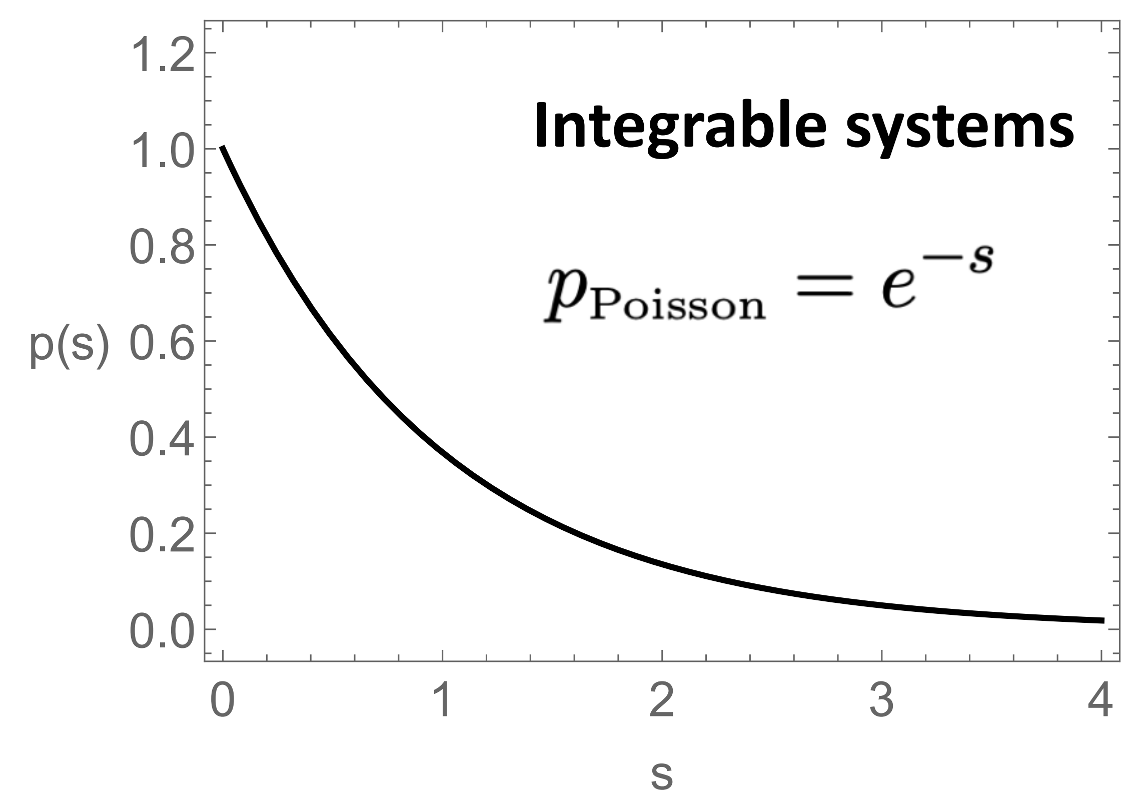




[J. Math. Phys. 3 (1962)]

[Phys. Rev. Lett. 52 (1984)]

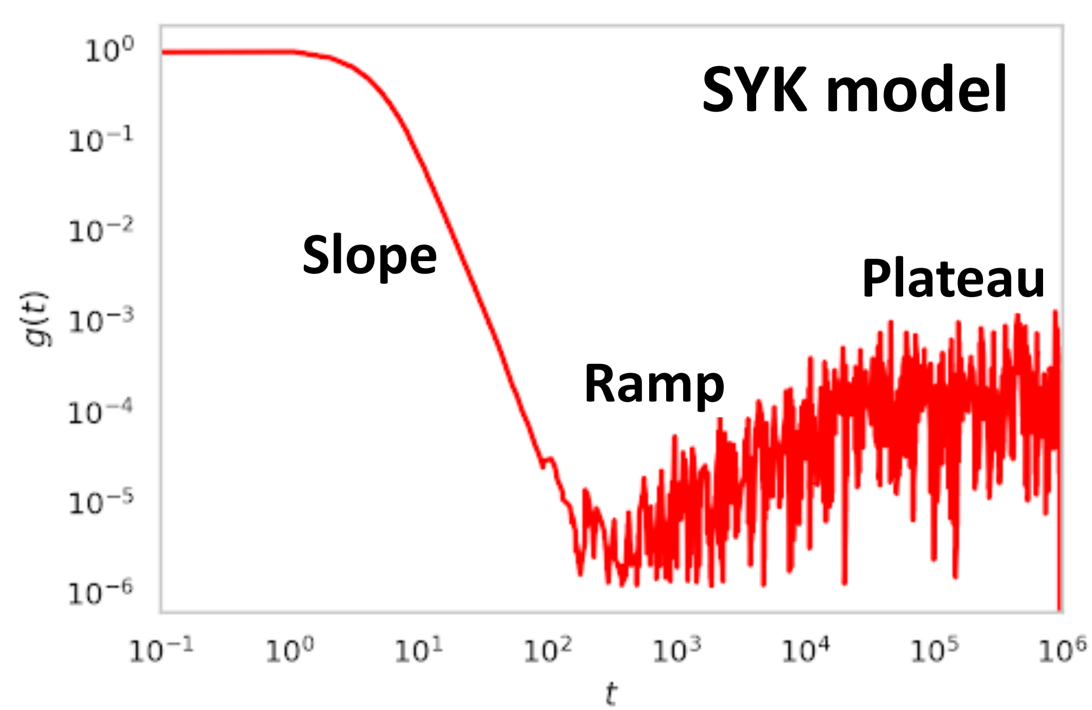
Bohigas-Giannoni-Schmit (BGS) Conjeture

- Spectral Form Factor (time-dependent characteristics of the spectrum)

inverse temperature

$$\mathrm{SFF} = \frac{|Z(\beta,t)|^2}{|Z(\beta,0)|^2}, \qquad Z(\beta,t) = \mathrm{Tr}\left[e^{-(\beta - it)H}\right]$$

: the hall mark of **chaotic systems** is the emergence of a "**linear**" ramp at late times.



[JHEP 05 (2017) 118]

Krylov Complexity

[Phys. Rev. D 106 (2022) 046007]

- **Krylov complexity of states** (new tool for probing quantum chaos)

Time-evolved state

Krylov basis

$$C(t) = \sum_n n|\psi_n(t)|^2, \qquad |\psi(t)\rangle = \sum_n \psi_n(t)|K_n\rangle$$

: It quantified the spread of a quantum state over the Krylov basis of given Hamiltonian

: A reference quantum state $|\psi(0)\rangle$ "spreads" and becomes complex

Hugo's Talk

: A complementary perspective to (time-dep) spectral measures (e.g., spectral form factor)

Krylov Complexity

[Phys. Rev. D 106 (2022) 046007]

- Krylov complexity of states (new tool for probing quantum chaos)

$$|\psi(0)\rangle = \frac{1}{\sqrt{Z(\beta, t=0)}} \sum_n e^{-\frac{\beta E_n}{2}} |n\rangle \otimes |n\rangle \,, \qquad H|n\rangle = E_n |n\rangle .$$

: For thermofield double states, Krylov complexity reveals a ramp-“**peak**”-slope-plateau

Krylov Complexity

[Phys. Rev. D 106 (2022) 046007]

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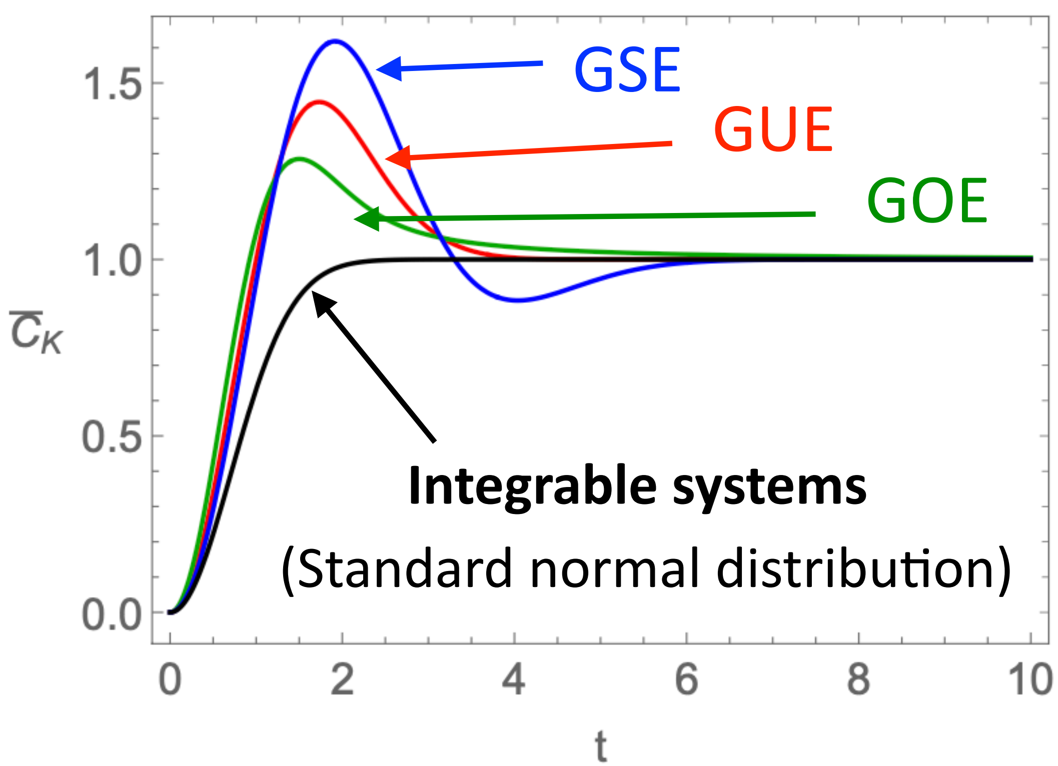


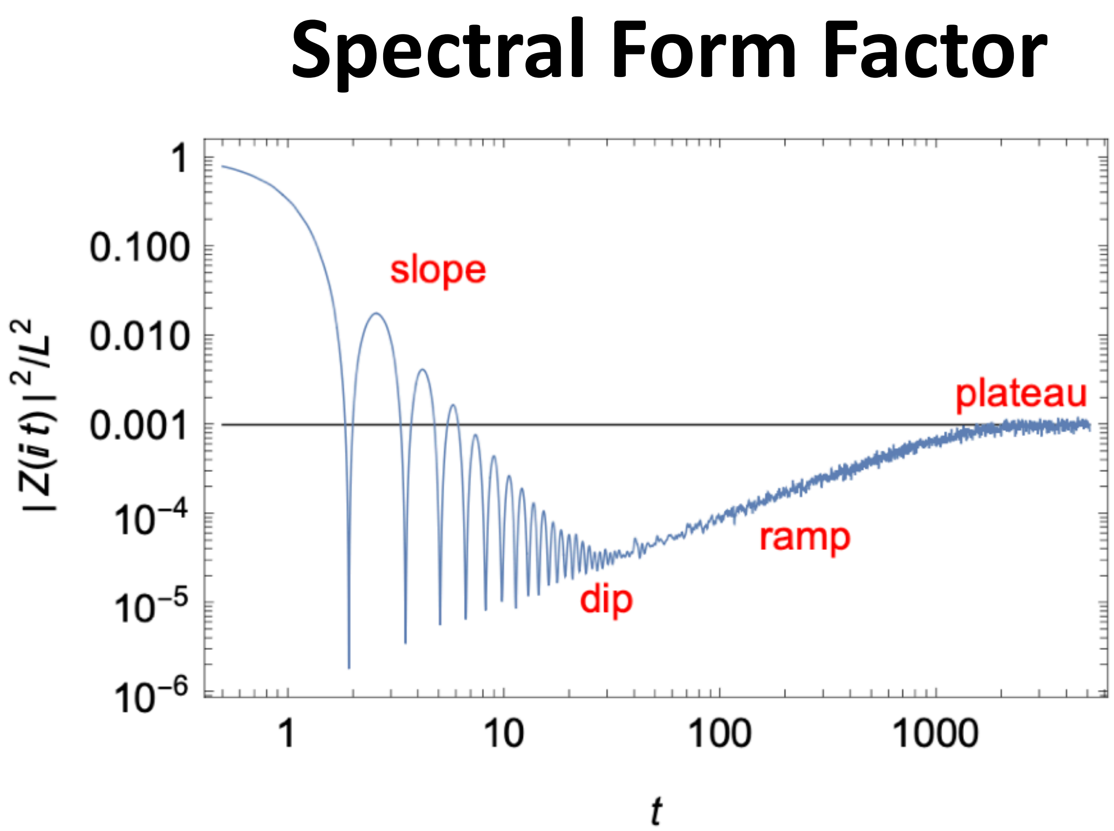
Peak is proposed as indicative of **chaotic** dynamics

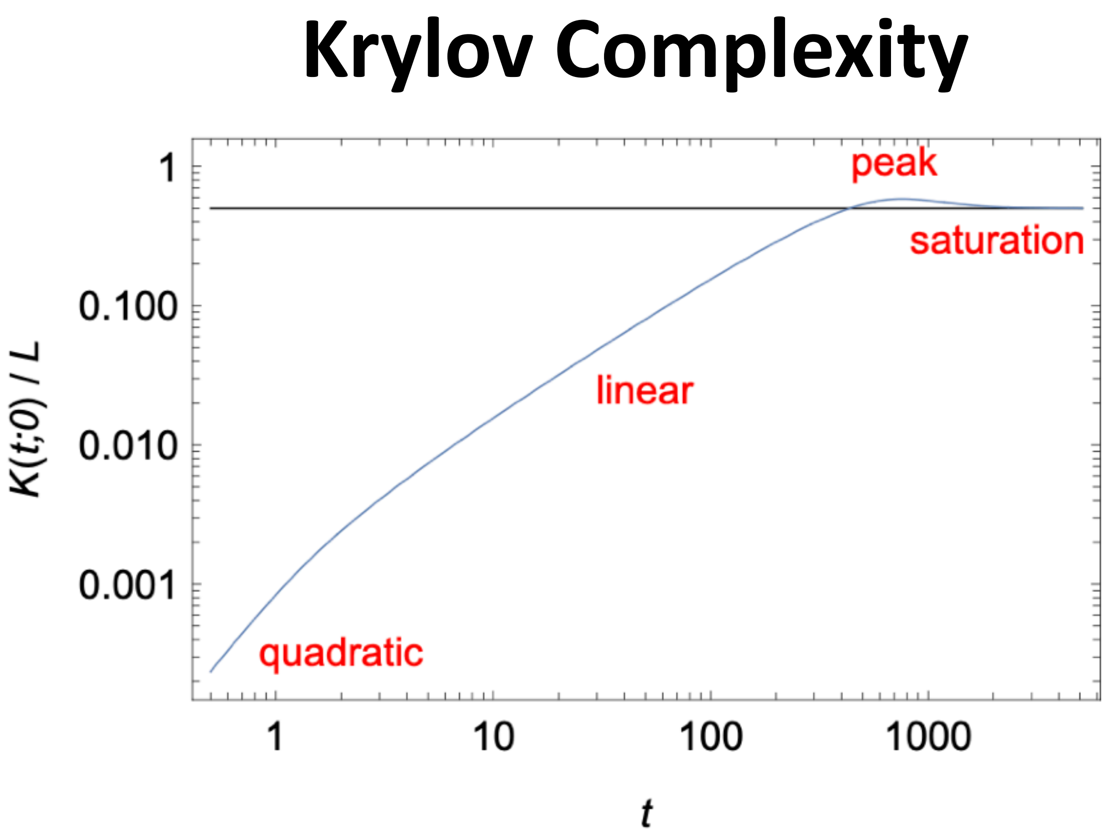
(Tested with diverse quantum mechanical models)

(RMT, SYK, Billiards, Spin-chain, etc...)

[JHEP 05 (2024) 337, JHEP 08 (2023) 176, JHEP 08 (2024) 241, ...]

[JHEP 08 (2023) 176]





: Four-stage behavior of Krylov complexity is analogous to the one from SFF

: For the maximally-entangled state, e.g., the TFD with $\beta = 0$

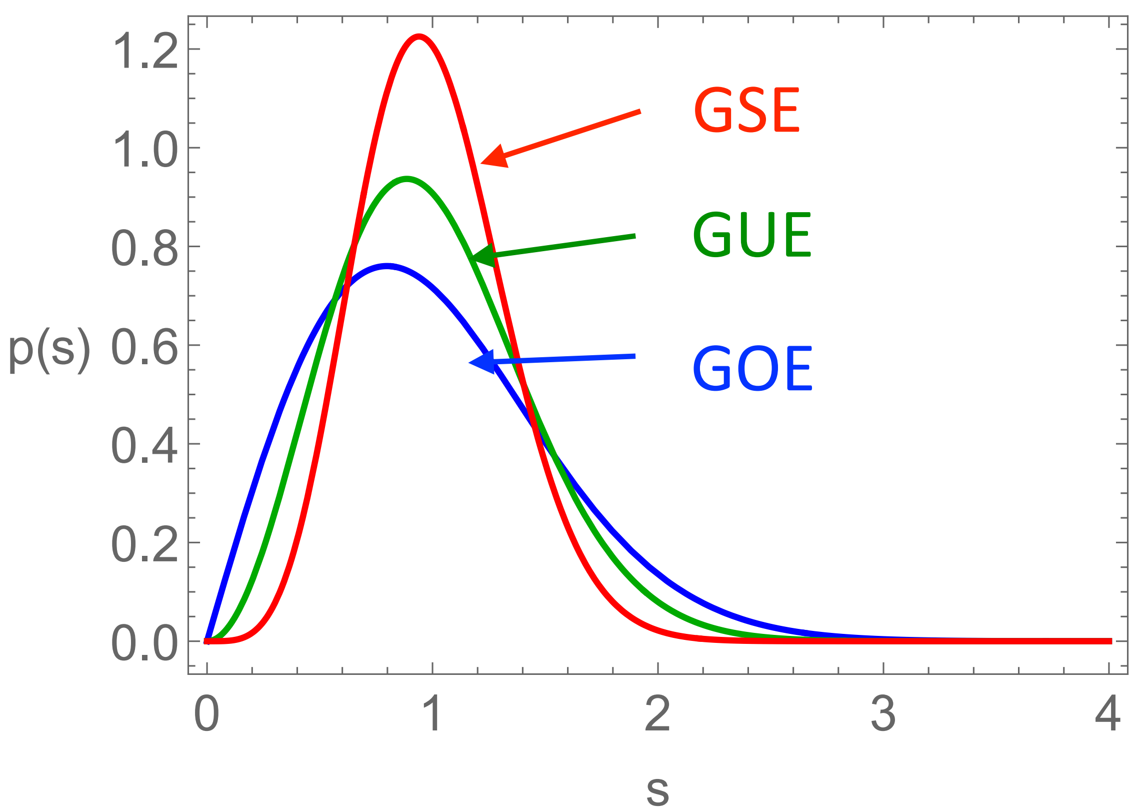
System size of H

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \mathrm{SFF}(t) = \frac{1}{1+2C(t=\infty)} \,, \qquad C(t=\infty) \approx \frac{d-1}{2} \,,$$

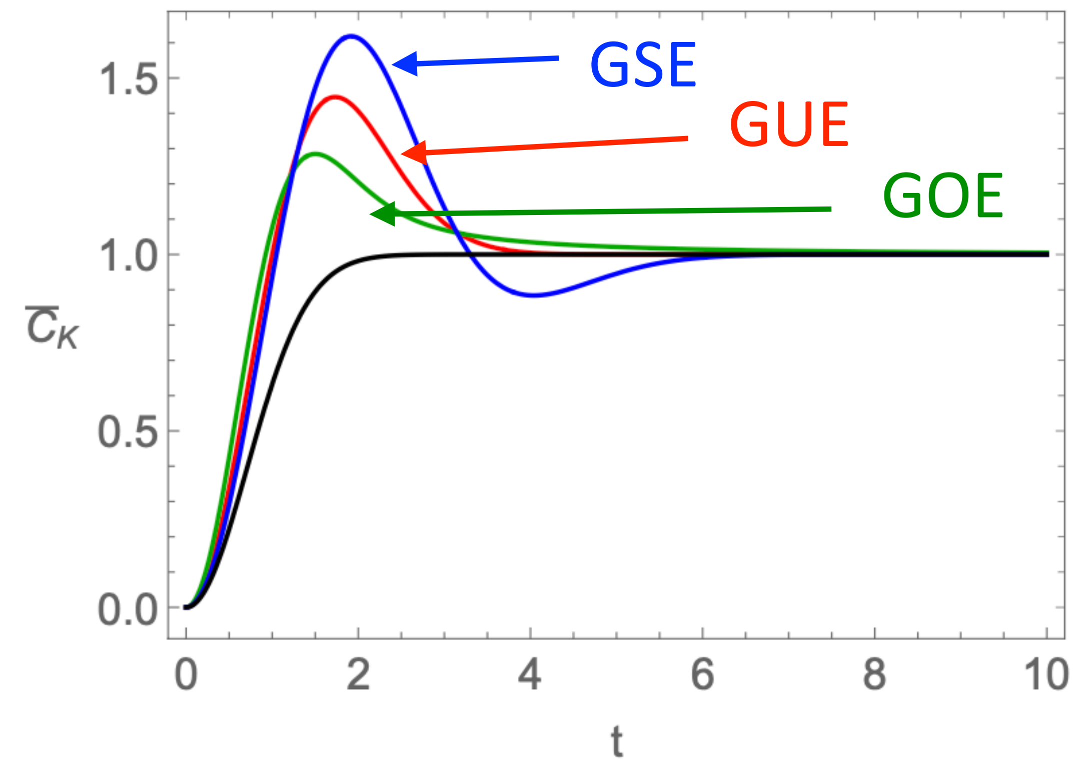
Quick Summary

All we need to evaluate **chaos diagnostics** is the **energy spectrum** of the system

1. Level spacing distribution



3. Krylov complexity



2. Spectral Form Factor: linear-ramp

Normal modes in Brickwall model

: energy spectrum from scalar/fermion fields

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2\mathrm{d}\varphi^2$$

$$r = \sqrt{\frac{1}{1-z}}$$

BTZ Black Hole

- Coordinate system

$$\mathrm{d}s^2 = -\frac{z}{1-z}\,\mathrm{d}t^2 + \frac{\mathrm{d}z^2}{4z(1-z)^2} + \frac{\mathrm{d}\varphi^2}{1-z}\,,$$

: It simplifies the computations of normal modes in Brickwall model

(it does not change any physics, of course)

: In this coordinate,

AdS boundary: $z \to 1$

Event horizon: $z \to 0$

Probe Scalar Field

$$(\Box - m_\Phi^2)\Phi = 0$$

$$\Phi = \phi(z)\, e^{-i\omega t} e^{iJ\varphi}$$

- Klein-Gordon Equation

$$\phi''(z) + \frac{\phi'(z)}{z} + \frac{J^2 z^2 + \omega^2 - z(J^2 + \omega^2 + m_\Phi^2)}{4z^2(1-z)^2}\phi(z) = 0\,.$$

: "J" is assumed as the quantum number in the Brickwall model

(interpreted as the angular quantum number) [Nucl. Phys. B 256 (1985) 727]

: "w" is the normal mode, interpreted as energy eigenvalues, as w(n, J)

↑ quantum numbers

: For simplicity, let us consider the massless case hereafter

Step 1

Probe Scalar Field

- Full solution

$$\phi(z) = e^{-\frac{\pi\omega}{2}} z^{-\frac{i\omega}{2}} \Big[C_1\, e^{\pi\omega}\, {}_2F_1\left(\frac{i(J-\omega)}{2}; \frac{-i(J+\omega)}{2}; 1-i\omega; z\right) + C_2\, z^{i\omega}\, {}_2F_1\left(\frac{-i(J-\omega)}{2}; \frac{i(J+\omega)}{2}; 1+i\omega; z\right) \Big]$$

with two undetermined coefficients (C_i) and hypergeometric functions (F)

Step 2

Probe Scalar Field

This whole this is

the leading term

- AdS boundary expansion ($z \to 1$)

$$\phi_{\text{bdry}}(z) \approx C_1 \frac{e^{\pi\omega}\,\Gamma\left[1-i\omega\right]}{\Gamma\left[1+\frac{i(J-\omega)}{2}\right]\Gamma\left[1-\frac{i(J+\omega)}{2}\right]} + C_2 \frac{\Gamma\left[1+i\omega\right]}{\Gamma\left[1-\frac{i(J-\omega)}{2}\right]\Gamma\left[1+\frac{i(J+\omega)}{2}\right]}$$

we impose the normalizability, $\phi_{\text{bdry}}(1) = 0,$ leading to

$$C_2 = -C_1\, e^{\pi\omega} \frac{\Gamma\left[1-\frac{i(J-\omega)}{2}\right]\Gamma\left[1+\frac{i(J+\omega)}{2}\right]\Gamma\left[1-i\omega\right]}{\Gamma\left[1+\frac{i(J-\omega)}{2}\right]\Gamma\left[1-\frac{i(J+\omega)}{2}\right]\Gamma\left[1+i\omega\right]}$$

setting the relationship between two undetermined coefficients (C_i)

Step 3

Probe Scalar Field

- Event horizon expansion ($z \to 0$)

$$\phi_{\rm hor}(z) \approx C_1 \left(P_1 \, z^{-\frac{i\omega}{2}} + Q_1 \, z^{\frac{i\omega}{2}} \right)$$

with

$$P_1 = 1\,, \qquad Q_1 = -\frac{\Gamma\left[1-\frac{i(J-\omega)}{2}\right]\Gamma\left[1+\frac{i(J+\omega)}{2}\right]\Gamma\left[1-i\omega\right]}{\Gamma\left[1+\frac{i(J-\omega)}{2}\right]\Gamma\left[1-\frac{i(J+\omega)}{2}\right]\Gamma\left[1+i\omega\right]}\,,$$

1) This is the combination of incoming and outgoing conditions

2) The functional form of P_1 and Q_1 is complicated in r-coordinate

Step 4

$$\phi_{\rm hor}(z) \approx C_1 \left(P_1 \, z^{-\frac{i\omega}{2}} + Q_1 \, z^{\frac{i\omega}{2}} \right)$$

Probe Scalar Field

- B.C at the stretched horizon near the event horizon $(z = z_0)$

$$\phi_{\rm hor}(z = z_0) = C_1 \left(P_1 \, z_0^{-\frac{i\omega}{2}} + Q_1 \, z_0^{\frac{i\omega}{2}} \right) =: \phi_0 \, = \mu_J \, e^{i\lambda_J \, \omega}$$

↑

Dirichlet boundary condition
with any constant value

Step 4

$$\phi_{\rm hor}(z) \approx C_1 \left(P_1\, z^{-\frac{i\omega}{2}} + Q_1\, z^{\frac{i\omega}{2}} \right)$$

Probe Scalar Field

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Dirichlet boundary condition with any constant value

1. A phase redefinition does not change any physics on normal modes

$$\lambda_J \to \lambda_J/\omega$$

2. Fixing the freedom makes the calculations simpler

$$C_1 Q_1 = 1$$

[JHEP 01 (2023) 153, JHEP 10 (2023) 231, JHEP 02 (2024) 049, ...]

Step 4

$$\phi_{\rm hor}(z) \approx C_1 \left(P_1 \, z^{-\frac{i\omega}{2}} + Q_1 \, z^{\frac{i\omega}{2}} \right)$$

Probe Scalar Field

- B.C at the stretched horizon near the event horizon $(z = z_0)$

$$\phi_{\rm hor}(z = z_0) = C_1 \left(P_1 \, z_0^{-\frac{i\omega}{2}} + Q_1 \, z_0^{\frac{i\omega}{2}} \right) =: \phi_0 = \mu_J \, e^{i\lambda_J \, \omega}$$

3. In general, P and Q are the complex numbers (given by equations)

$$P_1 = |P_1| e^{i\theta_\alpha} \, , \qquad Q_1 = |Q_1| e^{i\theta_\beta} \, .$$

Step 4

$$\phi_{\rm hor}(z) \approx C_1 \left(P_1\, z^{-\frac{i\omega}{2}} + Q_1\, z^{\frac{i\omega}{2}} \right)$$

Probe Scalar Field

- B.C at the stretched horizon near the event horizon $(z = z_0)$

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3. In general, P and Q are the complex numbers (given by equations)

$$P_1 = |P_1| e^{i\theta_\alpha}\,, \qquad Q_1 = |Q_1| e^{i\theta_\beta}\,.$$

Quantization Condition

Imaginary part: $\cos(\theta_\alpha - \theta_\beta) = \cos(2\lambda_J\, \omega)\,, \quad \sin(\theta_\alpha - \theta_\beta) = \sin(2\lambda_J\, \omega)$

⟶ θ are functions of (ω, J) **given by the equation of motion**

Step 4

$$\phi_{\rm hor}(z) \approx C_1 \left(P_1\, z^{-\frac{i\omega}{2}} + Q_1\, z^{\frac{i\omega}{2}}\right)$$

Probe Scalar Field

- B.C at the stretched horizon near the event horizon $(z = z_0)$

$$\phi_{\rm hor}(z=z_0) = C_1 \left(P_1\, z_0^{-\frac{i\omega}{2}} + Q_1\, z_0^{\frac{i\omega}{2}}\right) =: \phi_0 = \mu_J\, e^{i\lambda_J\,\omega}$$

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Quantization Condition

Imaginary part: $\cos(\theta_\alpha - \theta_\beta) = \cos(2\lambda_J\,\omega)\,, \quad \sin(\theta_\alpha - \theta_\beta) = \sin(2\lambda_J\,\omega)$

⟶ θ are functions of (ω, J) **given by the equation of motion**

free parameter ↓

⟶ This phase equation provides normal modes $\omega(n\,,J)$ with an integer n for given λ_J

Step 4

Probe Scalar Field

$$\phi_{\rm hor}(z) \approx C_1 \left(P_1\, z^{-\frac{i\omega}{2}} + Q_1\, z^{\frac{i\omega}{2}} \right)$$

$$\theta = {\rm Arg}\left[z_0^{i\omega} \right]$$

- B.C at the stretched horizon near the event horizon $(z = z_0)$

$$\phi_{\rm hor}(z = z_0) = C_1 \left(P_1\, z_0^{-\frac{i\omega}{2}} + Q_1\, z_0^{\frac{i\omega}{2}} \right) =: \phi_0 = \mu_J\, e^{i\lambda_J\, \omega}$$

$$\mu_J = 2\cos\left(\lambda_J\, \omega - \tfrac{\theta}{2}\right)$$

4. The free parameter is heuristically comparable to the position of the stretched horizon

λ_J

[JHEP 01 (2023) 153, JHEP 10 (2023) 231, JHEP 02 (2024) 049, ...]

⟶ Let us model λ_J as drawn from a **Gaussian distribution** with standard deviation σ_J

↑ **free parameter**

⟶ In the zero-variance limit: $\langle \lambda_J \rangle = \dfrac{1}{2} \log z_0 \;\; \to -\infty$

$\mu_J = 2$

as the stretched horizon approaches to the event horizon $z_0 \to 0$

Step 4

$$\phi_{\rm hor}(z) \approx C_1 \left(P_1\, z^{-\frac{i\omega}{2}} + Q_1\, z^{\frac{i\omega}{2}} \right)$$

Probe Scalar Field

$$\theta = \mathrm{Arg}\left[z_0^{i\omega}\right]$$

- B.C at the stretched horizon near the event horizon $(z = z_0)$

$$\phi_{\rm hor}(z = z_0) = C_1 \left(P_1\, z_0^{-\frac{i\omega}{2}} + Q_1\, z_0^{\frac{i\omega}{2}} \right) =: \phi_0 = \mu_J\, e^{i\lambda_J\, \omega}$$

$$\mu_J = 2\cos\left(\lambda_J\, \omega - \tfrac{\theta}{2}\right)$$

Long story short, from the phase equation, we determine the normal modes with

$$\langle \lambda_J \rangle = \frac{1}{2} \log z_0$$

the position of the stretched horizon

$$\sigma_J$$

the standard deviation

control parameter σ_0

$\longrightarrow$ $\langle \lambda_J \rangle = -10^4$

$$\sigma_J := \sigma_0,\ \sigma_0/J,\ \text{or}\ \sigma_0/\sqrt{J}$$

[JHEP 01 (2023) 153, JHEP 10 (2023) 231, JHEP 02 (2024) 049, ...]

[Fortsch. Phys. 57 (2009)]

Probe Fermion Field

- Dirac Equation

$$\left(\Gamma^M D_M - m_\Psi\right)\Psi = 0$$

$$\Psi = \begin{pmatrix} \psi_+(\rho) \\ \psi_-(\rho) \end{pmatrix} e^{-i\omega t} e^{iJ\varphi}$$

: It can be solved analytically in our z-coordinate

$$\psi_\pm(z) = \sqrt{\frac{(1\pm\sqrt{z})\sqrt{1-z}}{\sqrt{z}}}\,(\chi_1(z)\pm\chi_2(z))$$

$$\begin{aligned}\chi_1(z) = (z-1)^{-\frac{1}{4}}\, z^{-\frac{i\omega}{2}}\Bigg[&C_1\, z^{i\omega}\,{}_2F_1\left(\frac{1}{4}-\frac{i(J-\omega)}{2}; -\frac{1}{4}+\frac{i(J+\omega)}{2}; \frac{1}{2}+i\omega; z\right) \\ &+ i\,C_2\, e^{\pi\omega}\sqrt{z}\,{}_2F_1\left(\frac{1}{4}+\frac{i(J-\omega)}{2}; \frac{3}{4}-\frac{i(J+\omega)}{2}; \frac{3}{2}-i\omega; z\right)\Bigg],\end{aligned}$$

$$\chi_2(z) = \frac{2}{1-2i(J+\omega)}\left[2(z-1)\,z^{1/2}\,\chi_1'(z) + i(Jz+\omega)\,z^{-1/2}\,\chi_1(z)\right].$$

Probe Fermion Field

$$\psi_{\rm hor}(z) \approx C_1 \left(P_1 \, z^{-\frac{i\omega}{2}} + Q_1 \, z^{\frac{i\omega}{2}} \right)$$

- Near horizon expansion

$$P_1 = -\frac{\cosh(\pi\omega) - i\sinh(J\pi)}{\pi \, 2^{-2i\omega}} \frac{\Gamma\left[\frac{1}{2} - i(J+\omega)\right] \Gamma\left[\frac{1}{2} + i(J-\omega)\right] \Gamma\left[\frac{1}{2} + i\omega\right]}{\Gamma\left[\frac{1}{2} - i\omega\right]}, \qquad Q_1 = 1,$$

which is different from the scalar field one

- Quantization condition

$$\cos(\theta_\alpha - \theta_\beta) = \cos(2\lambda_J \, \omega), \quad \sin(\theta_\alpha - \theta_\beta) = \sin(2\lambda_J \, \omega)$$

Results

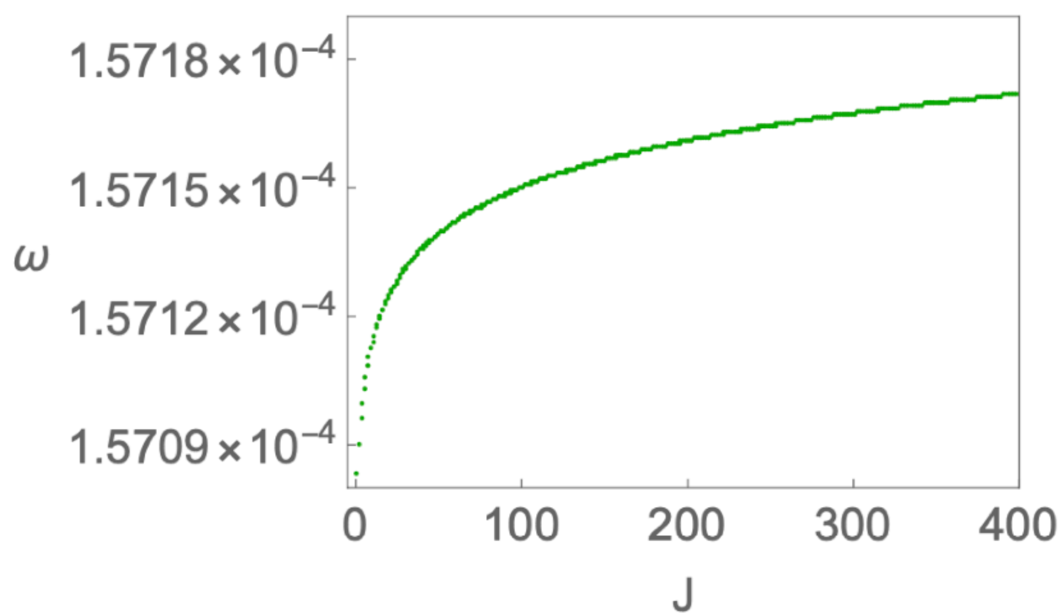
: Probe Scalar Field

$\langle \lambda_J \rangle = -10^4$

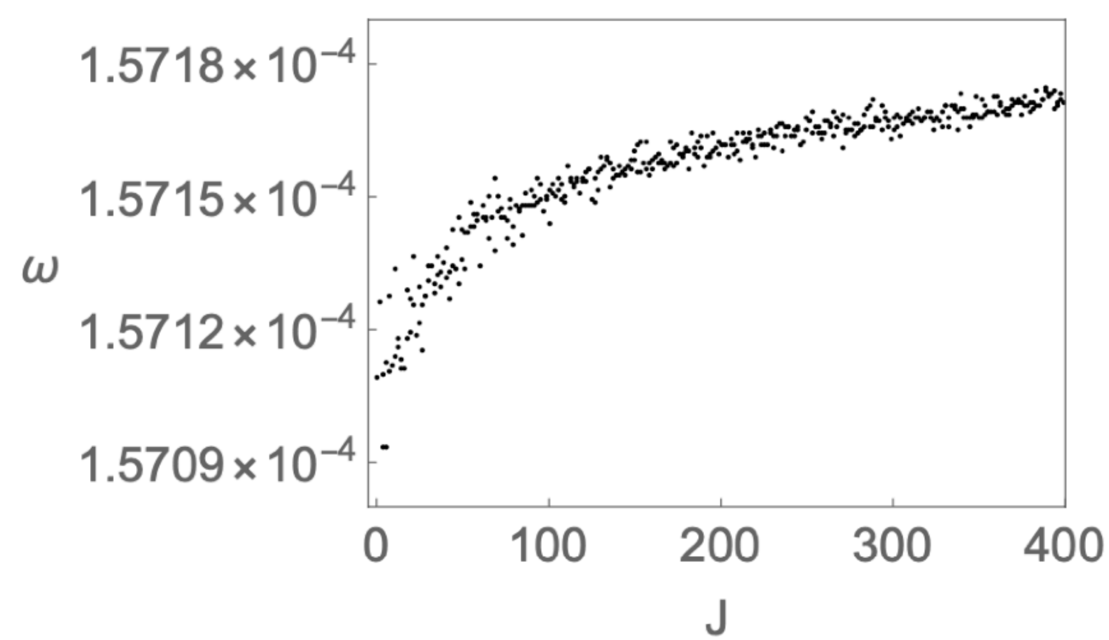
σ_0 (0~2)

Normal Modes

$\sigma_0 = 0.025$



$\sigma_0 = 2$

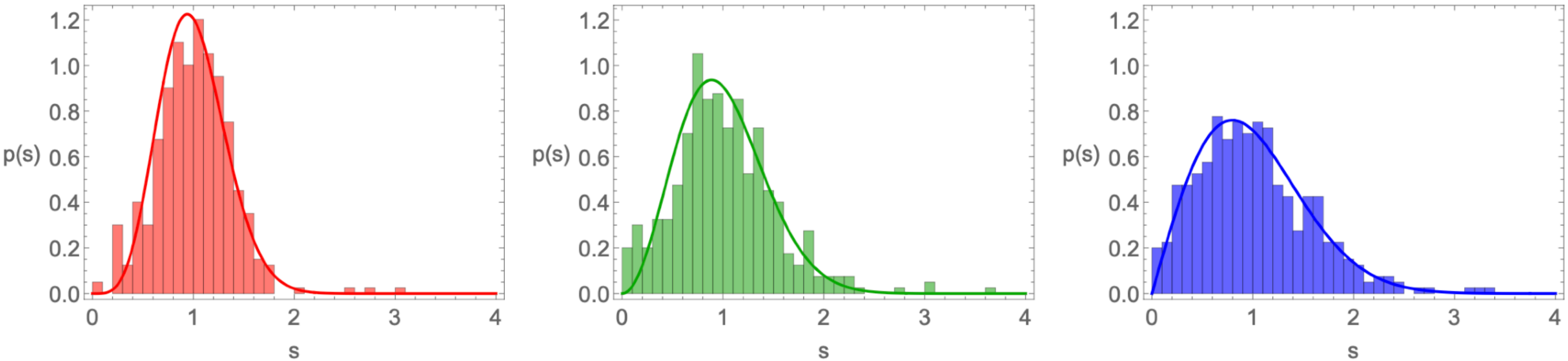


- It is n=0 mode, similar to higher levels

- It is symmetric in J < 0

- Erratic behavior as we increase the standard deviation

Level Spacing Distribution



(a) $\sigma_0 = 0.018$ (GSE) (b) $\sigma_0 = 0.025$ (GUE) (c) $\sigma_0 = 0.030$ (GOE)

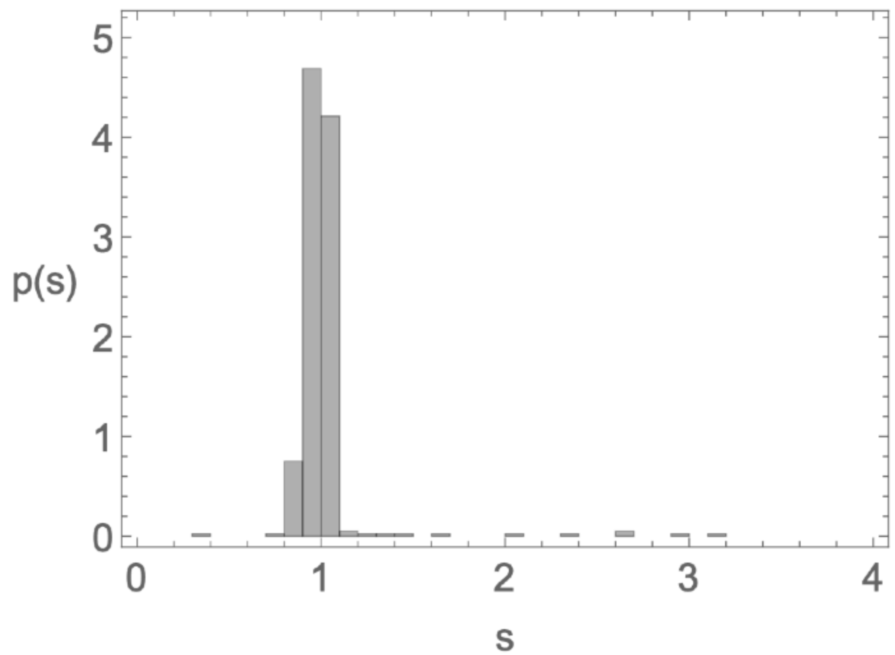
- **Intermediate value** of deviation, LSD follows the **Wigner-Dyson distribution** from RMT

(i.e., chaotic systems)

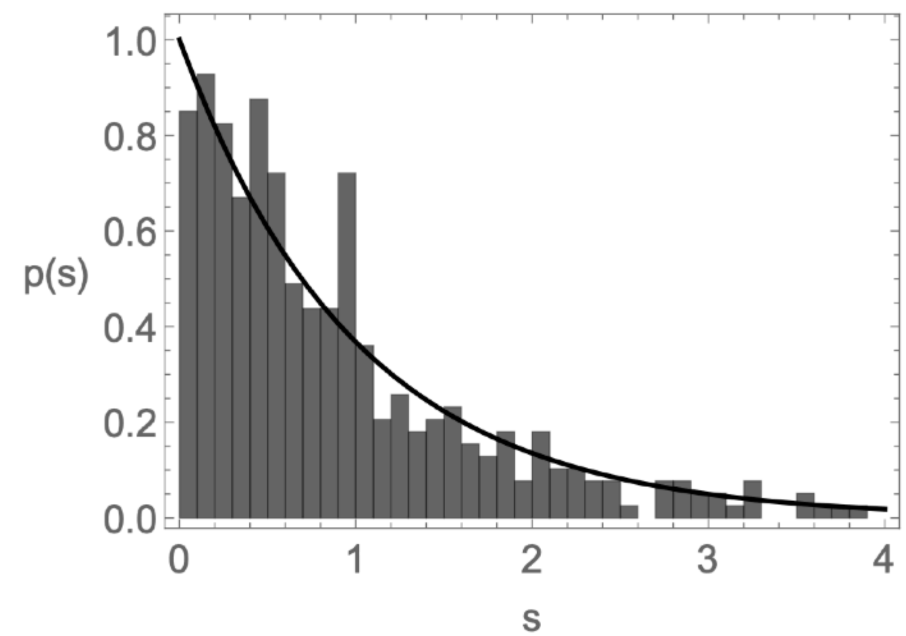
- GUE case was reported in previous literature

[JHEP 01 (2023) 153, JHEP 10 (2023) 231, JHEP 02 (2024) 049, …]

Level Spacing Distribution



(a) $\sigma_0 = 0$



(b) $\sigma_0 = 2$ (Poisson)

- **Extreme value** of deviation, LSD follows the **"harmonic-oscillator like"** distribution

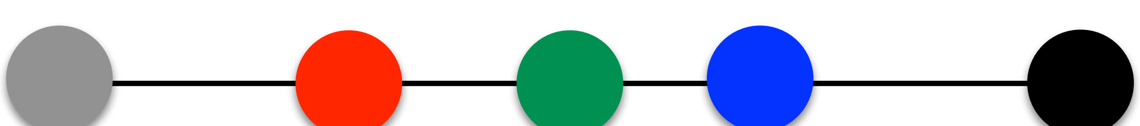
$E_n = \hbar\omega(n + \frac{1}{2})$

Or

Poisson distribution

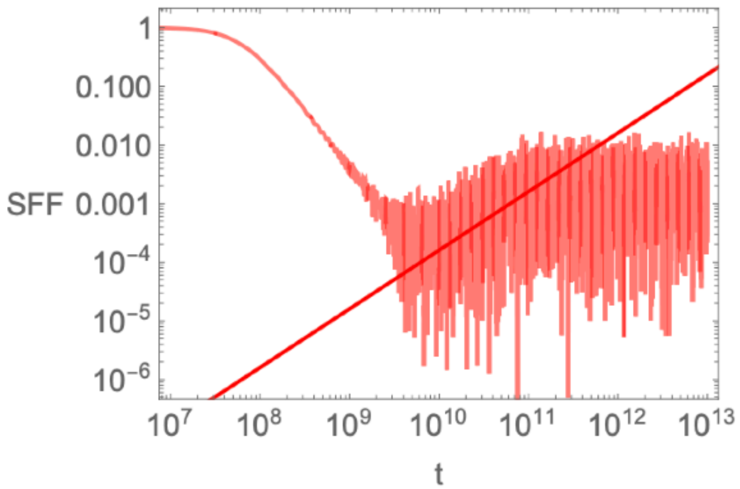
(i.e., Integrable systems!)

σ_0

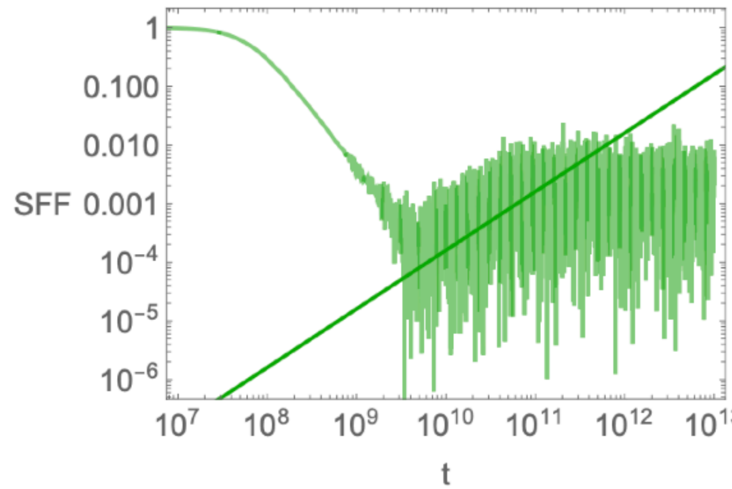


Integrable Chaos Integrable

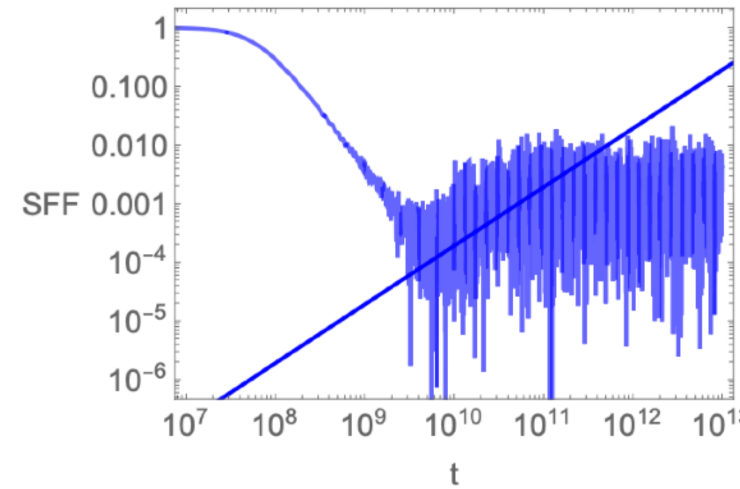
Spectral Form Factor



(a) $\sigma_0 = 0.018$ (GSE)

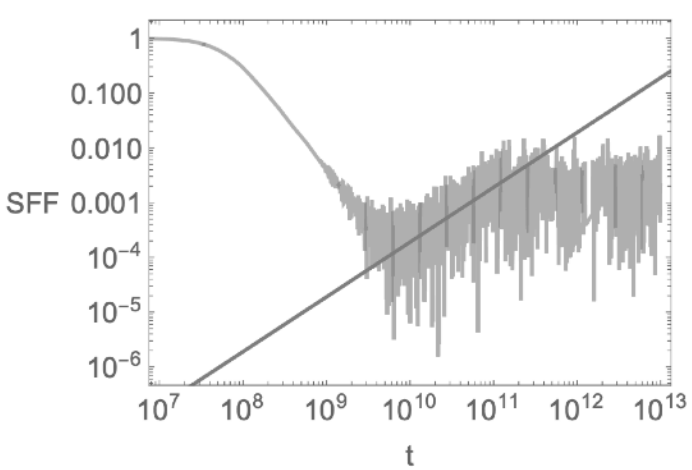


(b) $\sigma_0 = 0.025$ (GUE)

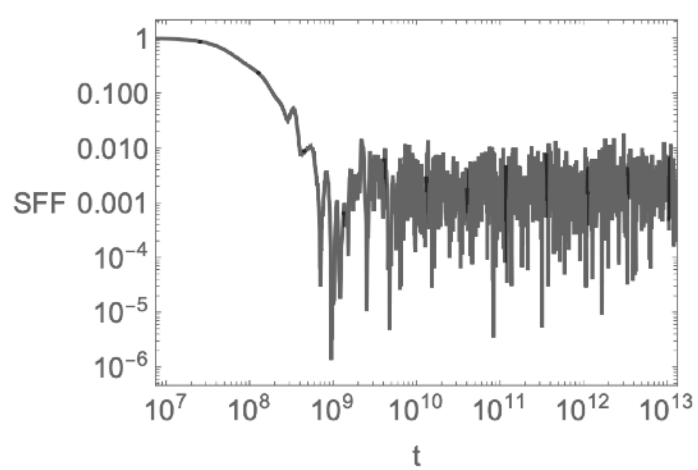


(c) $\sigma_0 = 0.030$ (GOE)

- **Intermediate value** of deviation, SFF exhibits the **linear-ramp** as in RMT



(a) $\sigma_0 = 0$



(b) $\sigma_0 = 2$ (Poisson)

Some **Integrable** system can show the **linear-ramp** in SFF

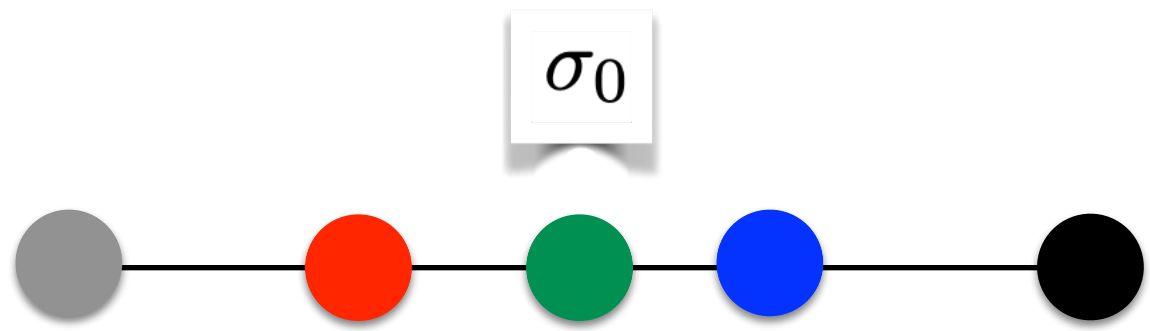
Toy model: $E_n \sim \log n$

Saddle-dominated scrambling

[JHEP 01 (2024) 172, JHEP 05 (2024) 137, ...]

- **Extreme value** of deviation, SFF can exhibit the **linear-ramp** when $\sigma_0 = 0$

Quick Summary



Integrable Chaos Integrable

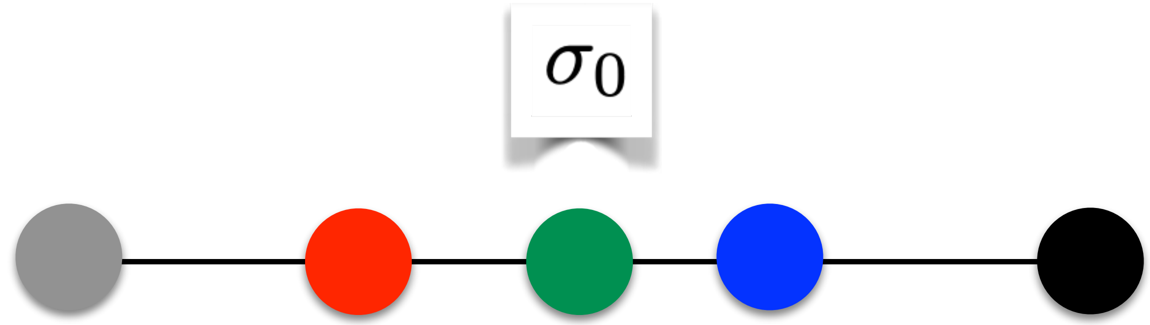
(Saddle-dominated scrambling) (RMT) (Poisson theory)

- “HO-like” LSD

- ramp in SFF

mimicking chaotic features
while remaining integrable

Quick Summary



Integrable | Chaos | Integrable

(Saddle-dominated scrambling) | (RMT) | (Poisson theory)

- “HO-like” LSD

- ramp in SFF

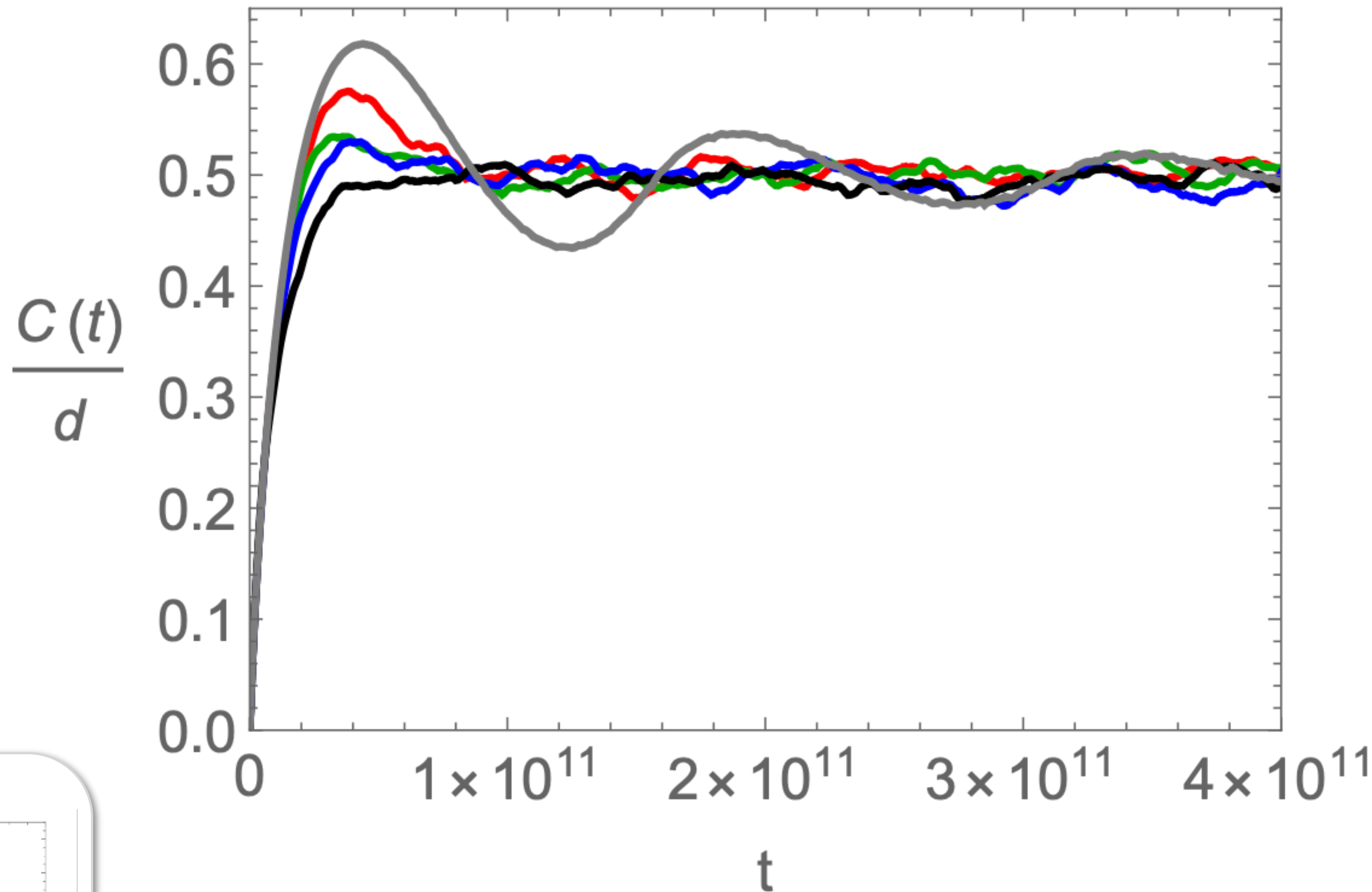
mimicking chaotic features
while remaining integrable

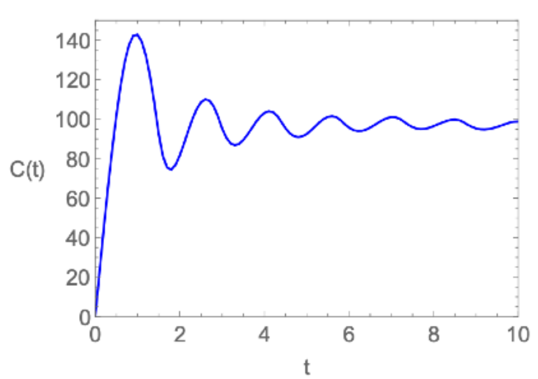
Q. **Krylov Complexity** can also provide the consistent results?

A. Yes.

Krylov Complexity

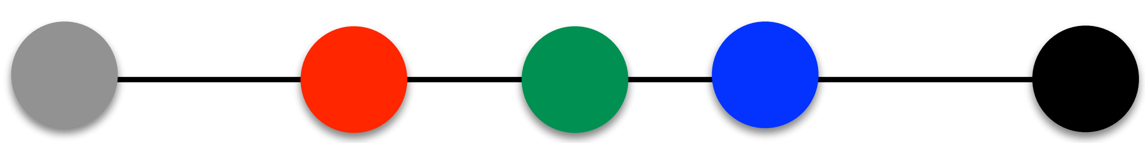
$$C(t = \infty) \approx \frac{d-1}{2}\,,$$





(Inverted H.O)

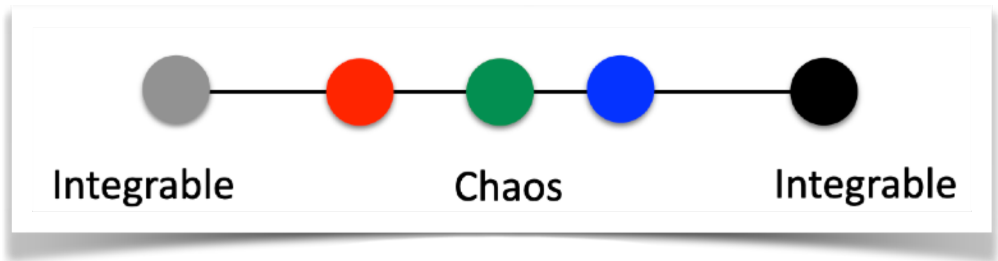
[JHEP 05 (2024) 137]



Integrable (peak) Chaos (peak) Integrable (x)

Some Remarks

- Mixed Phase



(Details in Appendix)

[arXiv: 2412.12301]

: LSD of the mixed phase is well described by the Wigner-Dyson and Brody Distribution.

: Dynamics of disappearance of the ramp in SFF, peak of Krylov complexity.

- Location of stretched horizon

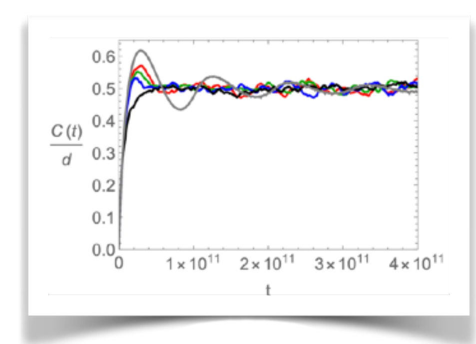
$$\langle \lambda_J \rangle = \frac{1}{2} \log z_0$$

(Details in Appendix)

[arXiv: 2412.12301]

: As noted in previous literature (GUE case), the signature of chaos (e.g., linear ramp) emerge when the stretched horizon is near the event horizon.

- Probe Fermion



: All results exhibits the same qualitative behavior as in the scalar field case.

: Numerical values of normal modes are different, but the underlying statistical behavior remain the same.

Summary

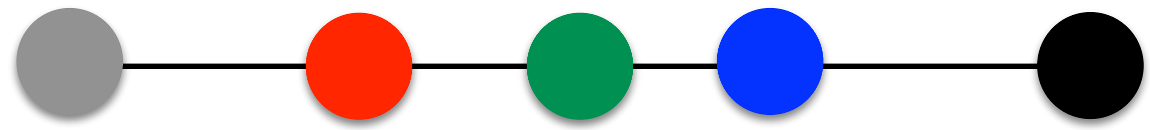
[arXiv: 2412.12301]

Energy Spectrum “Statistics”

&

Krylov Complexity

[arXiv: 2412.12301]



Integrable (Saddle-dom.) — Chaos (RMT) — Integrable (Poisson theory)

With Gaussian-distributed b.c.s on the stretched horizon, Brickwall model exhibit features consistent with RMT as well as the integrable features. **(across ensembles)**

- Dynamics of scalar/fermionic probe fields and normal modes thereof

- Wigner-Dyson Distribution, Linear-Ramp in SFF, Characteristic Peak in Krylov Complexity
  (Complexity with no-interior)

- Saddle-dominated scrambling and Poisson theory

- Interesting (one-parameter) gravitational toy models for { Quantum Chaos, Poisson Theory, Saddle-Dominated Scrambling }

  σ_0

Final Remarks

- Our analysis has adopted a phenomenological approach

: Need to better understand the Dirichlet wall b.c.s (still ad-hoc).

: Underlying conceptual origins of this phenomenon? Perspective of boundary CFT?

- Higher-dimensional analysis

Energy spectrum "statistics"?

: It may provide further insights.

: Normal modes of probe scalar field in 5-dim AdS (spherical symmetric metric) [arXiv: 2409.05519]

: Higher-dimensional hyperbolic black holes (Analytically solvable KG, Maxwell equations)

- dS black hole analysis [work in progress : HSJ, J. F. Pedraza, and J. M. Begines]

: Stretched horizon can be placed near the cosmological horizon. [JHAP 1 (2021) 1–22] (ds holography)?

: Insights into the chaotic properties and the phenomenon of hyperfast scrambling?

Final Remarks

- Relation with Quasi-Normal Modes?

Physical Review D

Brickwall, normal modes, and emerging thermality

Souvik Banerjee[1], Suman Das[2], Moritz Dorband[1], and Arnab Kundu[2]

In this paper, we demonstrate how black hole quasinormal modes can emerge from a Dirichlet brickwall model normal modes. We consider a probe scalar field in a Baños-Teitelboim-Zanelli geometry with a Dirichlet brickwall and demonstrate that as the wall approaches the event horizon, the corresponding poles in the retarded correlator become dense and yield an effective branch cut. The associated discontinuity of the correlator carries the information of the black hole quasinormal modes. We further demonstrate that a

- Open Quantum Systems?

[PRL, 61, 1899 (1988), PRL, 123, 254101 (2019), PRX, 10, 021019 (2020), ...]

: For the non-Hermitian Hamiltonian, the energy eigenvalues are complex number.

: We have different conjectures of the quantum chaos in OQS.

(Ginibre random matrix ensemble, Dissipative SFF, Complex spacing ratio, …)

: How Brickwall model can be extended and modified to capture these if any.

(complex normal modes?)

THANK YOU

Hyun-Sik Jeong

IFT Madrid

Quantum Gravity of Open Systems, 05 February 2025